Solutions to exam in MAT2400, Spring 2016

Problem 1: a) Since $|\arctan u| < \frac{\pi}{2}$ for all $u \in \mathbb{R}$, we have

$$\frac{|\arctan(nx)|}{n^2} < \frac{\frac{\pi}{2}}{n^2}$$

for all x. The series

$$\sum_{n=1}^{\infty} \frac{\frac{\pi}{2}}{n^2} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges, and hence by Weierstrass' M-test, the original series $\sum_{n=1}^{\infty} \frac{\arctan(nx)}{n^2}$ converges uniformly on all of $\mathbb{R}$. As f is the uniform limit of a sequence of continuous functions, it must be continuous.

b) Differentiating the series term by term, we get

$$\sum_{n=1}^{\infty} \frac{1}{n(1+n^2x^2)}$$

If this series converges uniformly in a neighborhood of x, we know by Corollary 4.3.6 that f is differentiable at x with

$$f'(x) = \sum_{n=1}^{\infty} \frac{1}{n(1+n^2x^2)}$$

Given an $x > 0$, we choose an a such that $0 < a < x$. Then for $u \in [a, \infty)$,

$$\frac{1}{n(1+n^2u^2)} \leq \frac{1}{n(1+n^2a^2)} \leq \frac{1}{a^2} \cdot \frac{1}{n^3}$$

As the series $\sum_{n=1}^{\infty} \frac{1}{a^2} \cdot \frac{1}{n^3} = \frac{1}{a^2} \sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, Weierstrass' M-test tells us that $\sum_{n=1}^{\infty} \frac{1}{n(1+n^2u^2)}$ converges uniformly on $[a, \infty)$, and hence f is differentiable at x with

$$f'(x) = \sum_{n=1}^{\infty} \frac{1}{n(1+n^2x^2)}$$

(By the way, a totally similar argument applies to $x < 0$; we just have to choose a such that $x < a < 0$ and work with the interval $(-\infty, a]$. On the other hand, one may show that the function is not differentiable at 0.)

Problem 2: a) As any bounded, closed set in $\mathbb{R}^n$ is compact, all closed balls $\overline{B}(a; r)$ in $\mathbb{R}^n$ are compact.

b) If $a \in X$, choose $r < |a|$. Then $\overline{B}(a; r)$ is compact as the set and the metric are the same as in $\mathbb{R}$, and the closed and bounded set $\overline{B}(a; r)$ is compact in $\mathbb{R}$. X is not complete as the Cauchy sequence $\{\frac{1}{n}\}$ does not converge in X.

Problem 3. a) Assume first that $x \in \bar{A}$. Then every ball $B(x, \frac{1}{n})$ contains an element a_n from A, and clearly the sequence $\{a_n\}$ converges to x. On the other hand, if there is a sequence $\{a_n\}$ from A converging to x, every ball $B(x, r)$ contains an element $a_n \in A$, and hence x cannot be an exterior point of A. This means that x is either an interior point or a boundary point, and in either case $x \in \bar{A}$.

b) Observe first that if $x \in \bar{A}$, then there is a sequence $\{a_n\}$ from A converging to x. As this sequence is also in $A \cup B$, we see that $x \in \overline{A \cup B}$. Hence $\bar{A} \subset \overline{A \cup B}$. A totally similar argument shows that $\bar{B} \subset \overline{A \cup B}$, and hence $\bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$.

To prove the opposite inclusion, assume that $x \in \overline{A \cup B}$. By a), there is a sequence $\{c_n\}$ from $A \cup B$ converging to x. This sequence must either have infinitely many terms from A or infinitely many terms from B (or both), say infinitely many from A. Let $\{c_{n_k}\}$ be the subsequence consisting of the terms that lie in A. As this is a sequence from A converging to x, we see that $x \in \bar{A}$. A similar argument shows that $x \in \bar{B}$ if infinitely many terms of $\{c_n\}$ belong to B. This means that if $x \in \overline{A \cup B}$, then $x \in \bar{A}$ or $x \in \bar{B}$, and hence $\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$. As we now have both inclusions, we see that

$$\overline{A \cup B} = \bar{A} \cup \bar{B}$$

To find an example of $\overline{A \cap B} \neq \bar{A} \cap \bar{B}$, we may let $X = \mathbb{R}$ and choose $A = (-\infty, 0)$, $B = (0, \infty)$. Then $\overline{A \cap B} = \bar{\emptyset} = \emptyset$ while $\bar{A} \cap \bar{B} = \{0\} \neq \emptyset$

Problem 4: a) Substituting $y = u - x$, we get $dy = -dx$ and

$$\begin{aligned}
(f * g)(u) &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(u-x) g(x)\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{u+\pi}^{u-\pi} f(y) g(u-y)\, (-dy) \\
&= \frac{1}{\sqrt{2\pi}} \int_{u-\pi}^{u+\pi} f(y) g(u-y)\, dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(y) g(u-y)\, dy = (g * f)(u)
\end{aligned}$$

where we have used the periodicity of the functions to get back to $[-\pi, \pi]$ as the interval of integration.

b) We have

$$\begin{aligned}
a_n b_n &= \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx \right) \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} g(y) e^{-iny}\, dy \right) \\
&= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \left(\int_{-\pi}^{\pi} f(x) e^{-inx}\, dx \right) g(y) e^{-iny}\, dy
\end{aligned}$$

$$= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \left(\int_{-\pi}^{\pi} f(x)g(y)e^{-in(x+y)}\, dx \right) dy$$

Introducing the new variable $u = x + y$ in the innermost integral, we get

$$a_n b_n = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \left(\int_{y-\pi}^{y+\pi} f(u-y)g(y)e^{-inu}\, du \right) dy$$

$$= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \left(\int_{-\pi}^{\pi} f(u-y)g(y)e^{-inu}\, du \right) dy$$

Changing the order of integration, we have

$$a_n b_n = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \left(\int_{-\pi}^{\pi} f(u-y)g(y)\, dy \right) e^{-inu}\, du$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(u-y)g(y)\, dy \right) e^{-inu}\, du$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} (f * g)(u) e^{-inu}\, du = c_n$$

c) Assume that there is a function k as in the problem, and let a_n be the n-th Fourier coefficients of k. Applying b) to k and e_n, we get

$$a_n \cdot 1 = 1$$

i.e. $a_n = 1$ for all n. This is impossible as $a_n \to \infty$ by the Riemann-Lebesgue lemma (or by Parseval's identity if you prefer).

Problem 5: a) We must show that $\|\cdot\|$ satisfies the three conditions for a norm:

(i) $\|\mathbf{x}\| \geq 0$ with equality if and only if $\mathbf{x} = 0$.

(ii) $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for all $\alpha \in \mathbb{R}$, $\mathbf{x} \in X$.

(iii) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in X$.

As $\|(x, y)\| = \max\{|x|, |y|\} = 0$ if and only if both x and y are 0, (i) is obvious. For (ii), note that if $|x| \geq |y|$, then $|\alpha||x| \geq |\alpha||y|$, and similarly that if $|y| \geq |x|$, then $|\alpha||y| \geq |\alpha||x|$. In either case,

$$\|\alpha\mathbf{x}\| = \max\{|\alpha||x|, \alpha||y|\} = |\alpha| \max\{|x|, |y|\} = |\alpha|\|\mathbf{x}\|$$

For (iii), let $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$. Then

$$|x_1 + y_1| \leq |x_1| + |y_1| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$$

and

$$|x_2 + y_2| \leq |x_2| + |y_2| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$$

Hence

$$\|\mathbf{x}+\mathbf{y}\| = \max\{|x_1+y_1|, |x_2+y_2|\} \leq \|\mathbf{x}\| + \|\mathbf{y}\|$$

b) Note that for $t > 0$,

$$\|\mathbf{a}+t\mathbf{r}\| = \|(1+t, 1+2t)\| = 1+2t$$

and hence $F(\mathbf{a}+t\mathbf{r}) = (1+2t)^2$. On the other hand, if $t < 0$, then

$$\|\mathbf{a}+t\mathbf{r}\| = \|(1+t, 1+2t)\| = 1+t$$

and hence $F(\mathbf{a}+t\mathbf{r}) = (1+t)^2$. If we try to compute the directional derivative $\mathbf{F}'(\mathbf{a};\mathbf{r}) = \lim_{t\to 0} \frac{\mathbf{F}(\mathbf{a}+t\mathbf{r})-\mathbf{F}(\mathbf{a})}{t}$ by taking one-sided limits, we get

$$\lim_{t\to 0^+} \frac{\mathbf{F}(\mathbf{a}+t\mathbf{r})-\mathbf{F}(\mathbf{a})}{t} = \lim_{t\to 0^+} \frac{(1+2t)^2-1}{t} = 4$$

and

$$\lim_{t\to 0^-} \frac{\mathbf{F}(\mathbf{a}+t\mathbf{r})-\mathbf{F}(\mathbf{a})}{t} = \lim_{t\to 0^-} \frac{(1+t)^2-1}{t} = 2$$

As the one-sided limits are unequal, the directional derivative $\mathbf{F}'(\mathbf{a},\mathbf{r})$ does not exist. Differentiable functions have directional derivatives, and hence $\mathbf{F}$ can not be differentiable at $\mathbf{a}$.

c) We first compute the directional derivatives to find a candidate for the derivative:

$$\mathbf{F}'(\mathbf{a};\mathbf{r}) = \lim_{t\to 0} \frac{\mathbf{F}(\mathbf{a}+t\mathbf{r})-\mathbf{F}(\mathbf{a})}{t} = \lim_{t\to 0} \frac{\|\mathbf{a}+t\mathbf{r}\|^2 - \|\mathbf{a}\|^2}{t}$$

$$\lim_{t\to 0} \frac{\langle \mathbf{a}+t\mathbf{r}, \mathbf{a}+t\mathbf{r}\rangle - \langle \mathbf{a}, \mathbf{a}\rangle}{t} = \lim_{t\to 0} \frac{2t\langle \mathbf{a}, \mathbf{r}\rangle + t^2\langle \mathbf{r}, \mathbf{r}\rangle}{t} = 2\langle \mathbf{a}, \mathbf{r}\rangle$$

This shows that $\mathbf{F}'(\mathbf{a})(\mathbf{r}) = 2\langle \mathbf{a}, \mathbf{r}\rangle$ is a promising candidate for the derivative. This function is obviously linear in $\mathbf{r}$, and since by Schwarz' inequality $|2\langle \mathbf{a}, \mathbf{r}\rangle| \leq 2\|\mathbf{a}\|\|\mathbf{r}\|$, it is a *bounded*, linear operator. It remains to show that

$$\sigma(\mathbf{r}) = \mathbf{F}(\mathbf{a}+\mathbf{r}) - \mathbf{F}(\mathbf{a}) - 2\langle \mathbf{a}, \mathbf{r}\rangle$$

goes to zero faster than $\mathbf{r}$. As

$$\sigma(\mathbf{r}) = \langle \mathbf{a}+\mathbf{r}, \mathbf{a}+\mathbf{r}\rangle - \langle \mathbf{a}, \mathbf{a}\rangle - 2\langle \mathbf{a}, \mathbf{r}\rangle = \langle \mathbf{r}, \mathbf{r}\rangle = \|\mathbf{r}\|^2$$

this is clearly the case, and hence $\mathbf{F}$ is differentiable with $\mathbf{F}'(\mathbf{a})(\mathbf{r}) = 2\langle \mathbf{a}, \mathbf{r}\rangle$.