MAT2400

Exam 2017

1 a) The fact that $u(x) \to a$ as $x \to -\infty$ means that $\forall\, \varepsilon > 0$ $\exists\, M \in \mathbb{R}$ such that $x < M \Rightarrow |u(x) - a| < \varepsilon$. Likewise, $\forall\, \varepsilon > 0$ $\exists\, N \in \mathbb{R}$ s.t. $x > N \Rightarrow |u(x) - b| < \varepsilon$.

Let $\varepsilon = 1$, $C = \min(|M|, |N|)$, and $B = |a| + |b| + 1$. If $x > C$ then $x > N$, so

$$|u(x)| \leq |u(x) - b| + |b| \leq 1 + |b| \leq B,$$

while if $x < -C$ then $x < M$, so

$$|u(x)| \leq |u(x) - a| + |a| \leq 1 + |a| \leq B.$$

Thus, $$|x| > C \Rightarrow |u(x)| \leq B. \qquad \blacksquare$$

(Note: I'm assuming the exam writer meant $|x| \geq C$ and not $x \geq C$.)

b) Let B, C be as in the previous problem, and let

$$M = \max\{|u(x)| : x \in [-C, C]\} + B$$

(this maximum exists since u is continuous on the compact set $[-C, C]$). If $|x| \leq C$ then

$$|u(x)| \leq \max\{|u(x)| : x \in [-C, C]\} \leq M,$$

while if $|x| > C$ then

$$|u(x)| \leq B \leq M.$$

Hence, $|u(x)| \leq M \quad \forall\, x \in \mathbb{R}$. $\blacksquare$

2 Let $\varepsilon > 0$ and let $N \in \mathbb{N}$ be s.t. $k \geq N \implies |u_n - \ell| < \varepsilon$. Then

$$|v_n - \ell| = \left| \left(\frac{1}{n+1} \sum_{k=n}^{2n} u_k \right) - \ell \underbrace{\left(\frac{1}{n+1} \sum_{k=n}^{2n} 1 \right)}_{=1} \right|$$

$$= \left| \frac{1}{n+1} \sum_{k=n}^{2n} (u_k - \ell) \right|$$

$$\leq \frac{1}{n+1} \sum_{k=n}^{2n} \underbrace{|u_k - \ell|}_{< \varepsilon}$$

$$< \frac{1}{n+1} \sum_{k=n}^{2n} \varepsilon = \varepsilon.$$

Hence, $v_n \to \ell$. $\blacksquare$

4 Let $X = C([-\pi,\pi], \mathbb{C})$ and

$Y = \{u \in X : u \text{ is Lipschitz}\}$.

a) We need to check that Y is closed under addition and multiplication by scalars.

If $u, v \in Y$ then $u+v \in X$, and if $\mathrm{Lip}(u)$ denotes the Lipschitz constant of u then

$$\left|\frac{(u+v)(x)-(u+v)(y)}{x-y}\right| = \left|\frac{u(x)-u(y)}{x-y} + \frac{v(x)-v(y)}{x-y}\right|$$

$$\leq \left|\frac{u(x)-u(y)}{x-y}\right| + \left|\frac{v(x)-v(y)}{x-y}\right| \leq \mathrm{Lip}(u)+\mathrm{Lip}(v).$$

Hence,

$$\mathrm{Lip}(u+v) \leq \mathrm{Lip}(u)+\mathrm{Lip}(v) < \infty,$$

so $u+v \in Y$.

If $u \in Y$ and $\alpha \in \mathbb{C}$ then

$$\left|\frac{\alpha u(x)-\alpha u(y)}{x-y}\right| = |\alpha|\left|\frac{u(x)-u(y)}{x-y}\right|$$

$\leq |\alpha| \operatorname{Lip}(u) < \infty,$

so $\alpha u \in Y$.

We conclude that Y is a linear subspace. ∎

b) Note first that if $M \in LIP(u)$ and $N \geq M$ then $N \in LIP(u)$, since

$$|u(x)-u(y)| \leq M|x-y| \leq N|x-y|.$$ Hence,

$LIP(u)$ is one of the intervals

$$(\operatorname{Lip}(u), \infty) \quad \text{or} \quad [\operatorname{Lip}(u), \infty),$$

so it suffices to check that $\operatorname{Lip}(u) \in LIP(u)$.

Let $\{M_n\}_n$ be a sequence in $LIP(u)$ s.t. $M_n \to \operatorname{Lip}(u)$ as $n \to \infty$. Then

$$|u(x)-u(y)| \leq M_n |x-y| \quad \forall\, n \in \mathbb{N},\ x,y \in [-\pi,\pi],$$

so taking $n \to \infty$ gives

$$|u(x)-u(y)| \leq \operatorname{Lip}(u)\, |x-y|,$$

and thus $\operatorname{Lip}(u) \in LIP(u)$. ∎

c) Clearly, $\|u\|_Y \geq 0 \ \forall u \in Y$, and if $\|u\|_Y = 0$ then both $|u(0)| = 0$ and $Lip(u) = 0$, so for any $x \in [-\pi, \pi]$,

$$|u(x)| = |u(x) - u(0) + \underbrace{u(0)}_{=0}| = |u(x) - u(0)|$$

$$\leq \underbrace{Lip(u)}_{=0} |x - 0| = 0$$

$$\Rightarrow \quad u = 0.$$

If $\alpha \in \mathbb{C}$ and $u \in Y$ then

$$\|\alpha u\|_Y = |\alpha u(0)| + Lip(\alpha u)$$

$$= |\alpha| |u(0)| + |\alpha| Lip(u) \quad \text{(as shown in (a))}$$

$$= |\alpha| \|u\|_Y.$$

Last, $\|u+v\|_Y = |u(0) + v(0)| + Lip(u+v)$

$$\leq |u(0)| + |v(0)| + Lip(u) + Lip(v)$$

(as shown in (a))

$$= \|u\|_Y + \|v\|_Y.$$

Hence, $\|\cdot\|_Y$ is a norm. $\blacksquare$

d) Let $u \in Y$, $x \in [-\pi, \pi]$. Then

$$
\begin{aligned}
|u(x)| &= |u(x) - u(0) + u(0)| \\
&\leq |u(x) - u(0)| + |u(0)| \\
&\leq \mathrm{Lip}(u) \underbrace{|x-0|}_{\leq \pi} + |u(0)| \\
&\leq \pi \|u\|_Y .
\end{aligned}
$$

Taking sup over $x \in [-\pi, \pi]$ yields

$$\|u\|_X \leq \pi \|u\|_Y . \qquad \blacksquare$$

I'm not sure about the question about the continuity property.

e) Let $u_n \in Y$ be given by

$$u_n(x) = \begin{cases} 0 & x \leq 0 \\ nx & 0 < x < 1/n \\ 1 & 1/n \leq x . \end{cases} \qquad (n \in \mathbb{N})$$

Then $\|u_n\|_X = 1$ and $\|u_n\|_Y = n$. Hence,

$$\|u_n\|_Y \leq n \|u_n\|_X ,$$

and since n can be arbitrarily large, there can be no single constant $C' > 0$ s.t. $\|u\|_Y \leq C' \|u\|_X \quad \forall \; u \in Y$.

$\blacksquare$