UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT2400 — Real Analysis

Day of examination: Friday June 01. 2018.

Examination hours: 14.30 – 18.30

This problem set consists of 4 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All problems (1a, 1b, 2a, 2b etc.) count for 10 points each. You have to explain all answers, and show enough details so that it is easy to follow your arguments. At the end of this document you will find some definitions that might be handy. You may answer the exam in either English or Norwegian.

Problem 1

Let X be the space $X = C([0,1], \mathbb{R})$ of continuous real valued functions on the interval $[0,1]$. We equip X with the sup-norm, *i.e.*, for $f \in X$ we set

$$\|f\| = \sup_{x \in [0,1]} \{|f(x)|\},$$

and we get the induced metric $d(f,g) = \|f - g\|$ for all $f, g \in X$.

(a) Let $L : X \to X$ be the map defined by

$$L(f)(x) = \int_0^x f(s)^3 ds.$$

Show that L has directional derivatives at each point $f \in X$.

(b) Show that L is differentiable at each point $f \in X$.

Solution:

(a) We have to compute $\lim_{t \to 0} \frac{L(f+tr) - L(f)}{t}$.

$$\begin{aligned}
\lim_{t \to 0} \frac{L(f+tr)(x) - L(f)(x)}{t} &= \lim_{t \to 0} \frac{1}{t} \int_0^x (f(s) + tr(s))^3 - f(s)^3 ds \\
&= \lim_{t \to 0} \frac{1}{t} \int_0^x f(s)^3 + 3f(s)^2 tr(s) + 3f(s)t^2 r(s)^2 \\
&\qquad + t^3 r(s)^3 - f(s)^3 ds \\
&= \lim_{t \to 0} \int_0^x 3f(s)^2 r(s) ds + \lim_{t \to 0} \int_0^x f(s)tr(s)^2 + t^2 r(s)^3 ds,
\end{aligned}$$

where the last integral converges to zero uniformly in x as $t \to 0$, and so the directional derivative is the function $\int_0^x 3f(s)^2 r(s) ds$.

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(b) The answer from (a) suggests that the derivative of L is the map $A : X \to X$ defined by

$$A(r)(x) = \int_0^x 3f(s)^2 r(s)ds.$$

We have that

$$\begin{aligned} A(\alpha r_1 + \beta r_2)(x) &= \int_0^x 3f(s)^2(\alpha r_1(s) + \beta r_2(s))ds \\ &= \alpha \int_0^x 3f(s)^2 r_1(s)ds + \beta \int_0^x 3f(s)^2 r_2(s)ds \\ &= \alpha A(r_1)(x) + \beta A(r_2)(x), \end{aligned}$$

which shows that A is linear. We also have that

$$|\int_0^x 3f(s)^2 r(s)ds| \leq \int_0^x 3|f(s)|^2|r(s)|ds \leq 3\|f\|^2 \cdot \|r\|,$$

which shows that A is bounded, since $\|f\|$ is a constant. Finally we have that

$$\begin{aligned} |\sigma_L(r)(x)| &= |L(f+r)(x) - L(f)(x) - A(r)(x)| \\ &= |\int_0^x (f(s) + r(s))^3 - f(s)^3 - 3f(s)^2 r(s)ds| \\ &= |\int_0^x f(s)^3 + 3f(s)^2 r(s) + 3f(s)r(s)^2 + r(s)^3 - f(s)^3 - 3f(s)^2 r(s)ds| \\ &= |\int_0^x 3f(s)r(s)^2 + r(s)^3 ds| \\ &\leq |\int_0^x |3f(s)r(s)^2 + r(s)^3|ds \leq K \cdot \|r\|^2, \end{aligned}$$

for $\|r\| \leq 1$, for some $K > 0$. So $\|\sigma_L(r)\| \leq K \cdot \|r\|^2$, which shows that $\|\sigma_L(r)\| \to 0$ sufficiently fast as $\|r\| \to 0$.

Problem 2

We let X be the space from Problem 1, and we define a map $T : X \to X$ by

$$T(f)(x) = \int_0^x \cos(\frac{f(s)}{2})ds.$$

(a) Show that the function $g(x) = \cos(\frac{x}{2})$ is Lipschitz continuous on $\mathbb{R}$.

(b) Show that T is a continuous mapping from X to X (We will take for granted that $T(f) \in X$ for each $f \in X$, so you don't have to show that.)

(c) Show that $T : X \to X$ has a unique fixed point.

Solution:

(a) We have that $|g'(x)| = (1/2)\sin(\frac{x}{2})$, which implies that $|g(x) - g(y)| \leq (1/2)|x - y|$ for all $x, y \in \mathbb{R}$.

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(b) For $u, v \in X$ we have that

$$\begin{aligned}
|T(u)(x) - T(v)(x)| &= |\int_0^x \cos(\frac{u(s)}{2}) - \cos(\frac{v(s)}{2}) ds| \\
&\le \int_0^x |\cos(\frac{u(s)}{2}) - \cos(\frac{v(s)}{2})| ds \\
&\le \int_0^x (1/2)|u(s) - v(s)| ds \\
&\le (1/2)\|u - v\|.
\end{aligned}$$

This shows that $\|T(u) - T(v)\| \le (1/2)\|u - v\|$, which shows that T is even Lipschitz continuous.

(c) From (b) we have that T is a contraction mapping, so this follows from Banach's Fixed Point Theorem.

Problem 3

Let Y be a non-empty set, and let d_1 and d_2 be metrics on Y. We let $d : Y \times Y \to \mathbb{R}$ be the function

$$d(x, y) = \sup\{d_1(x, y), d_2(x, y)\},$$

for all $x, y \in Y$.

(a) Show that d is a metric.

(b) Let $E = \{y_j\}_{j \in \mathbb{N}} \subset Y$ be a sequence of points, which is a Cauchy sequence with respect to both d_1 and d_2. Show that E is a Cauchy sequence with respect to d.

Solution:

(a) It is easy to see that $d(x, y) \ge 0$, $d(x, y) = 0$ if and only if $x = y$, and $d(x, y) = d(y, x)$. We have to check the triangle inequality, so we fix $x, y, z \in Y$. Then $d(x, z) = d_i(x, z)$ for $i = 1$ or $i = 2$ (or both). Assume that $i = 1$. Then

$$\begin{aligned}
d(x, z) = d_1(x, z) &\le d_1(x, y) + d_1(y, z) \\
&\le \sup\{d_1(x, y), d_2(x, y)\} + \sup\{d_1(y, z), d_2(y, z)\} \\
&= d(x, y) + d(y, z).
\end{aligned}$$

Repeat for $i = 2$.

(b) Fix $\epsilon > 0$. Then there exist $N_i \in \mathbb{N}, i = 1, 2$, such that

$$d_i(y_k, y_l) < \epsilon,$$

whenever $k, l \ge N_i$. Set $N = \max\{N_1, N_2\}$. Then

$$d(y_k, y_l) = \sup\{d_1(y_k, y_l), d_2(y_k, y_l)\} < \epsilon$$

whenever $k, l \ge N$, which shows that we indeed have a Cauchy sequence with respect to d.

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Problem 4

For $n \in \mathbb{N}$ let $f_n : [0, \frac{\pi}{2}] \to \mathbb{R}$ denote the function

$$f_n(x) = \cos(\frac{\pi}{2}(1 - \frac{\pi - 2x}{\pi})^n).$$

(a) Show that f_n converges pointwise to a function f as $n \to \infty$.

(b) Does f_n converge uniformly to f, *i.e.*, in sup-norm?

Solution:

(a) Note that $f_n(x) = \cos((\frac{\pi}{2})(\frac{2x}{\pi})^n)$. So if $\frac{2x}{\pi} < 1$, *i.e.*, if $x < \frac{\pi}{2}$, we see that $f_n(x) \to \cos(0) = 1$ is $n \to \infty$. On the other hand, if $x = \frac{\pi}{2}$ then $f_n(x) = \cos(\frac{\pi}{2}) = 0$ for all n. This shows pointwise convergence.

(b) We have that f_n does not converge uniformly. This is because the uniform limit of a sequence of continuous functions is continuous, but we have just seen that the pointwise limit is discontinuous.

Problem 5

Let X be the space from Problem 1. Let $F : X \to \mathbb{R}$ be a continuous map, and assume that $F(f) = 0$ whenever $f \in X$ is a polynomial. Show that $F(f) = 0$ for all $f \in X$.

Solution: Let $f \in X$ be an arbitrary function. By Weierstrass Approximation Theorem, there exists a sequence p_n of polynomials such that $p_n \to f$ uniformly as $n \to \infty$. Since F is continuous we have that

$$\lim_{n \to \infty} F(p_n) = F(f),$$

and since $F(p_n) = 0$ for all n, we have that $F(f) = 0$.

THE END

Some facts: Recall that if $L : X \to Y$ is a map between linear spaces, then the directional derivative $L'(f; r)$ at a point $f \in X$ in the direction $r \in X$ is given by

$$L'(f; r) = \lim_{t \to 0} \frac{L(f + tr) - L(f)}{t},$$

provided that the limit exists.

Recall that a function $g : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous if there exists a constant $K \geq 0$ such that $|g(x) - g(y)| \leq K \cdot |x - y|$ for all $x, y \in \mathbb{R}$.