UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT2400 — Real Analysis

Day of examination: 9 June 2021

Examination hours: 15:00 – 19:00

This problem set consists of 6 pages.

Appendices: None

Permitted aids: Any

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Note: There are in total 10 sub-problems, and you can get up to 10 points for each sub-problem, for a total of 100 points.

Updated 30 June 2021

Problem 1. (10 points)
Let (X,d) be the metric space $X = (0,1]$, $d(x,y) = |x-y|$, and let $T\colon X \to X$ be given by $T(x) = x/2$.

Show that T is a contraction. Does T have a fixed point? Justify your answer.

Solution: $d(T(x),T(y)) = |x/2 - y/2| = |x-y|/2 = d(x,y)/2$, so T is a contraction with contraction constant $1/2$.

T does not have a fixed point. If X were complete then we could apply Banach's fixed point theorem, but X isn't complete. The only solution to $x = T(x)$ is $x = 0$, but $0 \notin X$, so T does not have a fixed point.

Problem 2. (10 points)
Let (X,d) be a metric space and let $A \subseteq \mathbb{R}$ be closed. We define the metric space $C_b(X,A) = \big\{\text{all continuous, bounded } f\colon X \to A\big\}$, equipped with the supremum metric

$$\rho(f,g) = \sup_{x\in X} |f(x) - g(x)|. \tag{1}$$

(You do not need to show that this is a metric space.) Show that $C_b(X,A)$ is a closed subset of $C_b(X,\mathbb{R})$.

Solution: Let $\{f_n\}_n$ be a sequence in $C_b(X,A)$ converging to some $f \in C_b(X,\mathbb{R})$. We claim that $f \in C_b(X,A)$, that is, that $f(t) \in A$ for every $t \in X$. Indeed, for every $t \in X$, the sequence $\{f_n(t)\}_n$ is a convergent sequence in the closed set A, to the limit $f(t)$ also lies in A. This concludes the proof.

Alternatively: Let $f \in \partial C_b(X,A)$. We want to show that $f(t) \in A$ for every $t \in X$. Let $t \in X$. Since $f \in \partial C_b(X,A)$, there are for every

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$\varepsilon > 0$ functions $g \in B(f;\varepsilon) \cap C_b(X, A)$ and $h \in B(f;\varepsilon) \setminus C_b(X, A)$. In particular, $g(t) \in A$ and $|f(t) - g(t)| \leqslant \rho(f, g) < \varepsilon$. Hence, for every $\varepsilon > 0$, the set $B_{\mathbb{R}}(f(t);\varepsilon) \cap A$ is nonempty (since it contains the number $g(t)$), so $f(t) \in \bar{A}$. But A is closed, so $\bar{A} = A$, and $f(t) \in A$.

Problem 3. (20 points)
Let (X, d) be a metric space, and for every nonempty $E \subseteq X$ and $x \in X$, define

$$\operatorname{dist}(x, E) = \inf\{d(x, y) : y \in E\}. \tag{2}$$

(a) Show that if E is compact and nonempty, then there is some $z \in E$ such that $\operatorname{dist}(x, E) = d(x, z)$.

(b) Give an example of a metric space (X, d), a point $x \in X$ and a nonempty subset $E \subseteq X$ for which there is no such point $z \in E$.

Solution:

(a) Let $\{y_n\}_{n\in\mathbb{N}} \subseteq E$ be a minimizing sequence: $d(x, y_n) \to \operatorname{dist}(x, E)$ as $n \to \infty$. Since E is compact, there is some $z \in E$ and a subsequence $\{y_{n(k)}\}_{k\in\mathbb{N}}$ such that $y_{n(k)} \to z$ as $k \to \infty$. But then

$$\operatorname{dist}(x, E) = \lim_{n\to\infty} d(x, y_n)$$

(*if a sequence converges, then so does any subsequence*)

$$= \lim_{k\to\infty} d(x, y_{n(k)})$$

(*if a sequence converges, then so does its distance from a fixed point*)

$$= d(x, z)$$

Alternatively: Let $f : E \to \mathbb{R}$ be the function $f(y) = d(x, y)$. Then f is a continuous function on a compact set, so by the extreme value theorem it attains a minimum: there is a point $z \in E$ where $d(x, z) \leqslant d(x, y)$ for all $y \in E$, so $d(x, z) = \inf\{d(x, y) : y \in E\}$.

(b) Let $X = \mathbb{R}$, $E = (0, 1)$ and $x = 2$. Then $\operatorname{dist}(x, E) = \lim_{y\to1} d(x, y) = 1$, but $d(x, z) > 1$ for every $z \in E$.

Problem 4. (20 points)
For this problem, recall that a bounded linear operator A is *invertible* if it is bijective and its inverse A^{-1} is bounded.

Let $(X, \|\cdot\|)$ be a normed vector space and let $A : X \to X$ be an invertible bounded linear operator. Define $\|x\|_A = \|Ax\|$ for every $x \in X$.

(a) Show that $\|\cdot\|_A$ is a norm on X.

(b) Show that a sequence $\{x_n\}_n$ in X converges in the norm $\|\cdot\|$ if and only if it converges in the norm $\|\cdot\|_A$.

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Solution:

(a) Denote $C = \|A\|_{\mathcal{L}}$, so that $\|Ax\| \leqslant C\|x\|$ for all $x \in X$.

Well-defined $\|x\|_A = \|Ax\|$ is a well-defined number for every $x \in X$, so the function $\|\cdot\|_A$ is well-defined.

Positivity It is clear that $\|x\|_A = \|Ax\| \geqslant 0$ for all $x \in X$, and that $\|0\|_A = 0$ (since A is linear). If $\|x\|_A = 0$ then $\|Ax\| = 0$, so $Ax = 0$. Since A is bijective and A^{-1} is linear, we get $x = A^{-1}0 = 0$.

Homogeneity If $\alpha \in \mathbb{R}$ then $\|\alpha x\|_A = \|A(\alpha x)\| = \|\alpha Ax\| = |\alpha|\|Ax\| = |\alpha|\|x\|_A$, where we first used homogeneity of A and then homogeneity of $\|\cdot\|$.

Triangle inequality If $x, y \in X$ then $\|x+y\|_A = \|A(x+y)\| = \|Ax + Ay\| \leqslant \|Ax\| + \|Ay\| = \|x\|_A + \|y\|_A$, where we used additivity of A and then the triangle inequality for $\|\cdot\|$.

(b) If $\|x_n - x\| \to 0$ as $n \to \infty$ then

$$\|x_n - x\|_A = \|A(x_n - x)\| \leqslant \|A\|_{\mathcal{L}}\|x_n - x\| \to 0$$

as $n \to \infty$. Conversely, if $\|x_n - x\|_A \to 0$ as $n \to \infty$ then

$$\begin{aligned}\|x_n - x\| &= \|A^{-1}(A(x_n - x))\| \leqslant \|A^{-1}\|_{\mathcal{L}}\|A(x_n - x)\| \\ &= \|A^{-1}\|_{\mathcal{L}}\|x_n - x\|_A \to 0\end{aligned}$$

as $n \to \infty$.

Alternatively: If the two norms are equivalent then convergence in one norm is equivalent to convergence in the other. We have for every $x \in X$ that

$$\|x\|_A = \|Ax\| \leqslant \|A\|_{\mathcal{L}}\|x\|$$

and

$$\|x\| = \|A^{-1}Ax\| \leqslant \|A^{-1}\|_{\mathcal{L}}\|Ax\| = \|A^{-1}\|_{\mathcal{L}}\|x\|_A.$$

In other words, there are constants $c, C > 0$ such that $c\|x\| \leqslant \|x\|_A \leqslant C\|x\|$, so the norms are equivalent.

Problem 5. (10 points)
Let f be given by the series

$$f(x) = \sum_{n=1}^{\infty} n(x-1)^n, \qquad x \in \mathbb{R}. \tag{3}$$

Determine the set $D = \{x \in \mathbb{R} : f(x) \text{ converges}\}$. Compute the derivative f', and determine the corresponding set D' of points where the series for f' converges.

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Solution: We compute the series' radius of convergence:

$$R^{-1} = \limsup_{n\to\infty} n^{1/n} = \limsup_{n\to\infty} e^{\log n/n} = 1.$$

Hence, the series converges pointwise at all $x \in (1-R, 1+R) = (0,2)$. At $x = 2$ the series diverges: $f(2) = \sum_{n=1}^{\infty} n = \infty$. At $x = 0$ we get

$$f(0) = \sum_{n=1}^{\infty} n(-1)^n.$$

The partial sums of this series is

$$\sum_{n=1}^{N} n(-1)^n = \begin{cases} N/2 & \text{if } N \text{ is even} \\ (-N-1)/2 & \text{if } N \text{ is odd,} \end{cases}$$

which clearly does not converge. Hence,

$$D = (0,2).$$

By the theory of power series, f is real analytic in $(0,2)$, so in particular the series consisting of derivatives of each summand converges for all $x \in (0,2)$, and

$$f'(x) = \sum_{n=1}^{\infty} n^2(x-1)^{n-1} = 1 + \sum_{n=1}^{\infty}(n+1)^2(x-1)^n.$$

In particular, the set D' certainly contains $(0,2)$ as a subset. If $x = 0$ then the series reads $\sum_{n=1}^{\infty}(-1)^{n-1}n^2$, which does not converge, and if $x = 2$ then we get $\sum_{n=1}^{\infty} n^2 = \infty$. Hence,

$$D' = (0,2).$$

Problem 6. (10 points)
Let $f, g\colon [-\pi,\pi] \to \mathbb{C}$ be continuous functions satisfying

$$\int_{-\pi}^{\pi} f(x)e^{inx}\,dx = \int_{-\pi}^{\pi} g(x)e^{inx}\,dx \qquad \forall\ n \in \mathbb{Z}. \tag{4}$$

Show that $f = g$.

Solution: Let $h = f - g$, so that $\int h(x)e^{inx}\,dx = 0$ for all $n \in \mathbb{N}$, that is, the Fourier coefficients of h are all 0. Since h is continuous on $[-\pi,\pi]$, it follows from Féjer's theorem that the partial Fourier series of h converges to h uniformly in Césaro mean. But the partial Fourier series of h is 0, so h must be 0, that is, $f = g$.

Alternatively: Let $h = f - g$, so that $\int h(x)e^{inx}\,dx = 0$ for all $n \in \mathbb{N}$, that is, the Fourier coefficients of h are all 0. Since h is continuous on $[-\pi,\pi]$, it follows the theory of Fourier series that the partial Fourier series converges to h in L^2. But the partial Fourier series of h is 0, so h

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must be 0, that is, $f = g$.

Problem 7. (20 points)
Let $X = C_b(\mathbb{R},\mathbb{R})$, equipped with the supremum norm $\|f\|_\infty = \sup_{t\in\mathbb{R}} |f(t)|$. Define

$$F\colon X \to X, \qquad F(f)(t) = 2f(t)^2 - e^{f(t)-t^2} \qquad \forall\, t \in \mathbb{R}. \tag{5}$$

(a) Prove that F is Fréchet differentiable and show that $F'(f) = A$ for $f \in X$, where $A\colon X \to X$ is given by

$$A(r)(t) = 4r(t)f(t) - r(t)e^{f(t)-t^2} \qquad \forall\, t \in \mathbb{R},\ r \in X. \tag{6}$$

Hint: You might need the fact that $|e^s - 1 - s| \leqslant \frac{e}{2}s^2$ *for every number* $|s| \leqslant 1$. *This follows from Taylor expansion of the exponential function.*

(b) Let $\mathbb{1}\colon \mathbb{R} \to \mathbb{R}$ be the constant function $\mathbb{1}(t) = 1$ for all $t \in \mathbb{R}$. Prove that F is bijective in a neighbourhood of $\mathbb{1}$. Compute $\big(F^{-1}\big)'(F(\mathbb{1}))$.

Solution:

(a) It is clear that A is linear. Moreover,

$$\|A(r)\|_\infty \leqslant 4\|r\|_\infty\|f\|_\infty + \|r\|_\infty e^{\|f\|_\infty} < \infty.$$

It is bounded by the above estimate, with $\|A\|_{\mathcal{L}} \leqslant 4\|f\|_\infty^2 + e^{\|f\|_\infty}$.

We have, for every $f, r \in X$ with $\|r\|_\infty \leqslant 1$, that

$$\begin{aligned}\big|\big(F(f+r) - F(f) - A(r)\big)(t)\big| &= \Big|2r(t)^2 - e^{f(t)-t^2}\big(e^{r(t)} - 1 - r(t)\big)\Big| \\ &\leqslant 2|r(t)|^2 + e^{f(t)}\big|e^{r(t)} - 1 - r(t)\big|\Big| \\ &\leqslant 2\|r\|_\infty^2 + e^{\|f\|_\infty}\frac{e}{2}\|r\|_\infty^2.\end{aligned}$$

Hence,

$$\frac{\|F(f+r) - F(f) - A(r)\|_\infty}{\|r\|_\infty} \leqslant \|r\|_\infty\big(2 + \tfrac{1}{2}e^{1+\|f\|_\infty}\big) \to 0$$

as $r \to 0$. This proves the claim.

(b) The Fréchet derivative F' is continuous everywhere and $F'(\mathbb{1})(r)(t) = r(t)\big(4 - e^{1-t^2}\big)$. If $s \in X$ then $F'(\mathbb{1})(r) = s$ if and only if $r(t) = \frac{s(t)}{4-e^{1-t^2}}$, which is a continuous, bounded function. Hence, $F'(\mathbb{1})$ is bijective, and

$$\|F'(\mathbb{1})^{-1}(s)\|_\infty = \sup_{t\in\mathbb{R}} \frac{|s(t)|}{|4 - e^{1-t^2}|} \leqslant \frac{1}{4-e}\|s\|_\infty,$$

so $F'(\mathbb{1})^{-1}$ is also bounded, with $\|F'(\mathbb{1})^{-1}\|_{\mathcal{L}} \leqslant \frac{1}{4-e} < \infty$. Hence, by the inverse function theorem, there are neighbourhoods U of $\mathbb{1}$ and V

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of $F(\mathbb{1})$ such that $F\colon U \to V$ is bijective. Moreover,

$$(F^{-1})'(F(\mathbb{1}))(s)(t) = F'(\mathbb{1})^{-1}(s)(t) = \frac{s(t)}{4 - e^{1-t^2}} \qquad \forall\ t \in \mathbb{R},\ s \in X.$$