UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in:	MAT2400 — Real analysis
Day of examination:	Thursday, August 18th, 2022
Examination hours:	15.00 – 19.00

This problem set consists of 2 pages.

Appendices:	None
Permitted aids:	None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All items (item 1a), 1b), 2a), etc.) count 10 points each. If there is a problem you cannot solve, you may still use the result freely in later problems. At the end of the problem set you will find some information that might be useful along the way.

Problem 1. The map $F\colon C([0,1],\mathbb{R}) \to \mathbb{R}$ is given by

$$F(x) = \int_0^1 e^{-s} x(s)^2 \, ds.$$

a) (10 points) Find the directional derivative $F'(x;r)$.

b) (10 points) Show that F is differentiable and find the derivative $F'(x)(r)$.

Problem 2.

a) (10 points) Show that $\sum_{n=0}^{\infty} e^{-nx} = \frac{e^x}{e^x-1}$ for $x > 0$, and that the series converges uniformly on all intervals $[a, \infty)$ where $a > 0$.

b) (10 points) Show that for $b > a > 0$, we have

$$\sum_{n=1}^{\infty} \frac{e^{-na} - e^{-nb}}{n} = \ln(e^b - 1) - \ln(e^a - 1) + a - b.$$

Problem 3. Let X consist of all functions $f\colon \mathbb{N} \to \mathbb{R}$ such that the set $\{m \in \mathbb{N} \,|\, f(m) \neq 0\}$ is finite.

a) (10 points) Show that $\|f\| = \sum_{m=1}^{\infty} |f(m)|$ defines a norm on X. (You don't need to check that X is a vector space.)

b) (10 points) Define a sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions in X by

$$f_n(m) = \begin{cases} 2^{-m} & \text{if } m \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Show that $\{f_n\}$ is a Cauchy sequence and that $(X, \|\cdot\|)$ is not complete.

(Continued on page 2.)

Problem 4.

a) (10 points) Assume that K is a nonempty, closed, and bounded subset of $\mathbb{R}^d$, and that $\mathbf{a}$ is a point in $\mathbb{R}^d$. Show that there is a point $\mathbf{x} \in K$ such that $\|\mathbf{x} - \mathbf{a}\| \le \|\mathbf{y} - \mathbf{a}\|$ for all $\mathbf{y} \in K$ (hence $\mathbf{x}$ is the point in K nearest to $\mathbf{a}$).

b) (10 points) Show that the conclusion in a) still holds if we remove the assumption that K is bounded (i.e., if we only assume that $K \subseteq \mathbb{R}^d$ is nonempty and closed).

Problem 5. Let $X = C([0,1], \mathbb{R})$ with the supremum norm:

$$\|f\| = \sup\{|f(x)| \,:\, x \in [0,1]\}.$$

Define the functions $f_n \in X$ by

$$f_n(x) = \begin{cases} nx & \text{if } 0 \le x \le \frac{1}{n} \\ 1 & \text{otherwise.} \end{cases}$$

a) (10 points) Make a sketch of f_n and show that

$$A = \{f_n \,:\, n \in \mathbb{N}\}$$

is a closed and bounded subset of X.

b) (10 points) Show that A is not compact.

GOOD LUCK!

Some information you might find useful

Sum of a convergent, geometric series:

$$\sum_{n=0}^{\infty} a_0 r^n = \frac{a_0}{1-r} \qquad \text{when } |r| < 1.$$

Weierstrass' M-test: Assume that $\sum_{n=0}^{\infty} M_n$ converges, and that $|v_n(x)| \le M_n$ for all n and all $x \in A$. Then $\sum_{n=0}^{\infty} v_n(x)$ converges uniformly on A.

Axioms for norms:

(i) $\|\mathbf{u}\| \ge 0$ with equality if and only if $\mathbf{u} = \mathbf{0}$.

(ii) $\|\alpha \mathbf{u}\| = |\alpha| \|\mathbf{u}\|$.

(iii) $\|\mathbf{u} + \mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$.

Definition of directional derivatives:

$$\mathbf{F}'(\mathbf{a}; \mathbf{r}) = \lim_{t \to 0} \frac{\mathbf{F}(\mathbf{a} + t\mathbf{r}) - \mathbf{F}(\mathbf{a})}{t}.$$

THE END