Solutions to exam in MAT2400, Spring 2022

Problem 1. The real Fourier series is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(nx) + a_n \sin(nx))$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx.$$

Since the functions $f(x)\cos(nx)$ are even and the functions $f(x)\sin(nx)$ are odd, we get by symmetry that

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos(nx)\,dx$$

$$b_n = 0$$

We first compute

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} 1\,dx = \frac{2}{\pi}\cdot\frac{\pi}{2} = 1.$$

For $n \geq 1$, we get

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos(nx)\,dx = \frac{2}{\pi}\left[\frac{\sin(nx)}{n}\right]_0^{\frac{\pi}{2}} = \frac{2}{n\pi}\sin\left(n\frac{\pi}{2}\right).$$

Observe that if n is even, then $\sin\left(n\frac{\pi}{2}\right)$ is 0, and if n is odd, then $\sin\left(n\frac{\pi}{2}\right)$ is 1 and -1 every second time starting at 1. Hence

$$a_{2n+1} = \frac{2}{(2n+1)\pi}(-1)^n.$$

This means that the Fourier series of f is

$$\frac{1}{2} + \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\cos[(2n+1)x].$$

b) Since f is differentiable at 0, Dini's Test (or one of its corollaries) tells us that $f(0)$ equals the sum of the Fourier series at 0:

$$f(0) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\cos[(2n+1)0],$$

i.e.

$$1 = \frac{1}{2} + \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}.$$

Collecting terms and multiplying by $\frac{\pi}{2}$, we get

$$\frac{\pi}{4} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Problem 2. a) By definition

$$F'(x;r) = \lim_{t\to 0}\frac{F(x+tr)-F(x)}{t}$$

$$= \lim_{t\to 0}\frac{(x(0)+tr(0))(x(1)+tr(1)) - x(0)x(1)}{t}$$

$$= \lim_{t\to 0}\frac{x(0)x(1)+tx(0)r(1)+tr(0)x(1)+t^2r(0)r(1)-x(0)x(1)}{t}$$

$$= \lim_{t\to 0}\big(x(0)r(1)+r(0)x(1)+tr(0)r(1)\big)$$

$$= x(0)r(1)+r(0)x(1).$$

b) We know that if F is differentiable, then $F'(x)(r) = F'(x;r) = x(0)r(1) + r(0)x(1)$, and we only have to check that $F'(x;r)$ satisfies the conditions of a derivative. If we write $A(r)$ for $F'(x;r)$, we first have to check that A is linear:

$$A(\alpha r+\beta s) = x(0)\big(\alpha r(1)+\beta s(1)\big) + \big(\alpha r(0)+\beta s(0)\big)x(1)$$

$$= \alpha\big(x(0)r(1)+r(0)x(1)\big) + \beta\big(x(0)s(1)) + s(0)x(1)\big) = \alpha A(r) + \beta A(s).$$

Next we check that A is bounded:

$$|A(r)| = |x(0)r(1)+r(0)x(1)| \leq |x(0)||r(1))| + |r(0)||x(1)|$$

$$\leq \|x\|\|r\| + \|r\|\|x\| = 2\|x\|\|r\|.$$

Finally, we must show that

$$\sigma(r) = F(x+r) - F(x) - A(r)$$

goes to 0 faster than r. We have

$$|\sigma(r)| = |(x(0)+r(0))(x(1)+r(1)) - x(0)x(1) - \big(x(0)r(1)+r(0)x(1))|$$

$$= |r(0)r(1)| \leq \|r\|^2$$

which clearly goes to 0 faster than r. Hence we have proved that F is differentiable with

$$F'(x)(r) = x(0)r(1) + r(0)x(1)$$

Alternative solution: We may also solve b) by using the product rule in Proposition 6.1.8: If we put $G(x) = x(0)$ and $H(x) = x(1)$, we get $F(x) = G(x)H(x)$ and

$$F'(x)(r) = G'(x)(r)H(x) + G(x)H'(x)(r).$$

As G and H are linear maps, they are their own derivatives, and hence $G'(x)(r) = G(r) = r(0)$ and $H'(x)(r) = H(r) = r(1)$. This gives

$$F'(x)(r) = G'(x)(r)H(x) + G(x)H'(x)(r) = r(0)x(1) + x(0)r(1).$$

It is also possible to use the Chain Rule to solve the problem.

Problem 3. Since $f(x) = a$ is *almost solvable*, there is for each $n \in \mathbb{N}$ an $x_n \in X$ such that $|f(x_n) - a| < \frac{1}{n}$. Since X is compact, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to a point $x_0 \in X$. By construction, $f(x_{n_k}) \to a$, and since f is continuous, $f(x_{n_k}) \to f(x_0)$. Hence $f(x_0) = a$.

A counterexample in the noncompact case is to let $X = (0, 1)$, $f(x) = x$, and $a = 0$. Then $f(x) = a$ is *almost solvable*, but there is no $x \in X$ such that $f(x) = a$.

Alternative solution: Observe that since f is continuous, so is $g(x) = |f(x) - a|$. By the Extreme Value Theorem, g has a minimum point x_0. Since g is non-negative, $g(x_0) \geq 0$, and since for every $\epsilon > 0$ there is an x such that $g(x) < \epsilon$, we must have $g(x_0) = 0$, i.e. $f(x_0) = a$.

Problem 4. a) By Bessel's inequality

$$0 = \|\mathbf{u} - \mathbf{u}\|^2 = \|\sum_{n=1}^{\infty} \alpha_n \mathbf{e}_n - \sum_{n=1}^{\infty} \beta_n \mathbf{e}_n\|^2 = \|\sum_{n=1}^{\infty} (\alpha_n - \beta_n)\mathbf{e}_n\|^2 \geq \sum_{n=0}^{\infty} (\alpha_n - \beta_n)^2$$

which implies that $(\alpha_n - \beta_n)^2 = 0$ for all n, and hence $\alpha_n = \beta_n$.

Alternative solution: Since $\mathbf{u} = \sum_{n=1}^{\infty} \alpha_n \mathbf{e}_n$, we have

$$\langle \mathbf{u}, \mathbf{e}_i \rangle = \langle \sum_{n=1}^{\infty} \alpha_n \mathbf{e}_n, \mathbf{e}_i \rangle = \sum_{n=1}^{\infty} \alpha_n \langle \mathbf{e}_n, \mathbf{e}_i \rangle = \alpha_i$$

Similarly, since $\mathbf{u} = \sum_{n=1}^{\infty} \beta_n \mathbf{e}_n$, we have

$$\langle \mathbf{u}, \mathbf{e}_i \rangle = \langle \sum_{n=1}^{\infty} \beta_n \mathbf{e}_n, \mathbf{e}_i \rangle = \sum_{n=1}^{\infty} \beta_n \langle \mathbf{e}_n, \mathbf{e}_i \rangle = \beta_i,$$

and hence we must have $\alpha_i = \beta_i$ for all $i \in \mathbb{N}$.

b) The sequence $\{\mathbf{e}_n\}_{n \in \mathbb{N}}$ can fail to be a basis for H in two ways: Either there is an element $\mathbf{u} \in H$ such that $\mathbf{u} \neq \sum_{n=1}^{\infty} \alpha_n \mathbf{e}_n$ for *all* sequences $\{\alpha_n\}$, or there is an element $\mathbf{u} \in H$ which can be written as a linear combination of the $\mathbf{e}_n$'s i two different ways: $\mathbf{u} = \sum_{n=1}^{\infty} \alpha_n \mathbf{e}_n = \sum_{n=1}^{\infty} \beta_n \mathbf{e}_n$. By a) the latter cannot happen in the present situation, and hence we are left with the first possibility that $\mathbf{u} \neq \sum_{n=1}^{\infty} \alpha_n \mathbf{e}_n$ for *all* sequences $\{\alpha_n\}$, and in particular for the sequence $\alpha_n = \langle \mathbf{u}, \mathbf{e}_n \rangle$.

c) Since H is complete, the series $\sum_{n=1}^{\infty} \alpha_n \mathbf{e}_n$ will converge if the partial sums $\mathbf{s}_k = \sum_{n=1}^{k} \alpha_n \mathbf{e}_n$ form a Cauchy sequence. Note that since $\sum_{n=0}^{\infty} |\alpha_n|^2 \leq \|\mathbf{u}\|^2$ by Bessel's inequality, the partial sums $S_k = \sum_{n=0}^{k} |\alpha_n|^2$ converge and hence form a Cauchy sequence. Given an $\epsilon > 0$, we can thus find a N such that for $k, m \geq N$, we have $\|S_m - S_k\| < \epsilon^2$. Hence (assuming $k \leq m$):

$$\|s_m - s_k\| = \| \sum_{n=k+1}^{m} \alpha_n \mathbf{e}_n \| = \left(\sum_{n=k+1}^{m} |\alpha_n|^2 \right)^{\frac{1}{2}} = \|S_m - S_k\|^{\frac{1}{2}} < \epsilon.$$

This means that the partial sums $\mathbf{s}_k = \sum_{n=1}^{k} \alpha_n \mathbf{e}_n$ form a Cauchy sequence and hence converge to an element $\mathbf{v}$. In other words, $\mathbf{v} = \sum_{n=1}^{\infty} \alpha_n \mathbf{e}_n$.

The "Fourier coefficients" β_m of $\mathbf{v}$ with respect to $\{\mathbf{e}_n\}_{n \in \mathbb{N}}$ are given by

$$\beta_m = \langle \mathbf{v}, \mathbf{e}_m \rangle = \langle \sum_{n=1}^{\infty} \alpha_n \mathbf{e}_n, \mathbf{e}_m \rangle = \sum_{n=1}^{\infty} \alpha_n \langle \mathbf{e}_n, \mathbf{e}_m \rangle = \alpha_m,$$

which shows that $\mathbf{u}$ and $\mathbf{v}$ have the same Fourier coefficients.

Alternative solution: The existence of a $\mathbf{v}$ such that $\mathbf{v} = \sum_{n=1}^{\infty} \alpha_n \mathbf{e}_n$ follows immediately from Proposition 5.3.11.

Problem 5. a) Using $\epsilon = 1$, we see that there is an $N \in \mathbb{N}$ such that $|f(x)| = |f(x) - 0| < 1$ when $x \geq N$. Since the interval $[0, N]$ is compact and f is continuous, the Extreme Value Theorem tells us that $|f|$ has a maximum value M on $[0, N]$. This means that $|f(x)| \leq \max\{M, 1\}$ for all x.

b) Observe first that by a), $\|f\| = \sup\{|f(x)| : x \in [0, \infty)\}$ is finite, and we only need to check the three properties of a norm:

(i) $\|f\| \geq 0$ with equality if and only if $f = 0$.

(ii) $\|\alpha f\| = |\alpha| \|f\|$.

(iii) $\|f + g\| \leq \|f\| + \|g\|$.

(i) By definition, $\|f\| \geq 0$ and $\|0\| = 0$. If $f \neq 0$, there is an a such that $f(a) \neq 0$, and hence

$$\|f\| = \sup\{|f(x)| : x \in [0, \infty)\} \geq |f(a)| > 0.$$

(ii) We have

$$\|\alpha f\| = \sup\{|\alpha f(x)| : x \in [0, \infty)\} = \sup\{|\alpha||f(x)| : x \in [0, \infty)\}$$

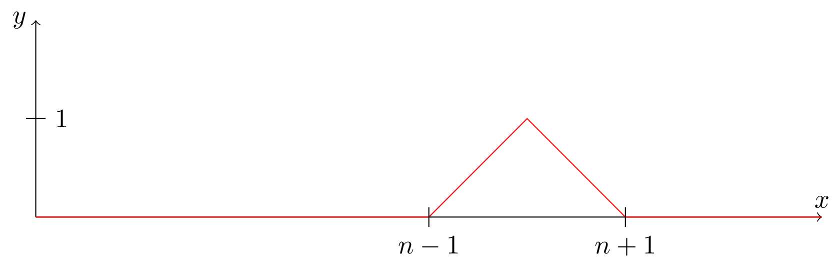
$$= |\alpha| \sup\{|f(x)| : x \in [0, \infty)\} = |\alpha| \|f\|.$$

(iii) We have

$$\|f + g\| = \sup\{|f(x) + g(x)| : x \in [0, \infty)\} \leq \sup\{|f(x)| + |g(x)| : x \in [0, \infty)\}$$

$$\leq \sup\{|f(x)| : x \in [0, \infty)\} + \sup\{|g(x)| : x \in [0, \infty)\} = \|f\| + \|g\|.$$

c) The figure shows the graph of e_n in red.



As $|e_n(x) - e_m(x)| \leq 1$ for all x and $|e_n(n) - e_m(n)| = 1$ when $n \neq m$, we have $\|e_n - e_m\| = 1$.

To show that B is not compact, it suffices to find a sequence in B that doesn't have a convergent subsequence. If we choose $\{e_n\}$ as our sequence, we see that for any subsequence $\{e_{n_k}\}$, we will have $\|e_{n_k} - e_{n_m}\| = 1$ when $k \neq m$. Hence $\{e_{n_k}\}$ is not a Cauchy sequence and cannot converge.

d) By Theorem 4.6.2, we know that the space $Y = C_b([0, \infty), \mathbb{R})$ of all bounded, continuous functions $f\colon [0, \infty) \to \mathbb{R}$ is complete in the supremum norm/metric, and by a) our space X is a subspace of Y. Since any closed subspace of a complete space is complete (Proposition 3.4.4), it suffices to prove that X is closed, and to prove that X is closed, it suffices to show that $X^c = Y \setminus X$ is open. To this end, chose a g in X^c. Since g is not in X, $g(x)$ does not converge to 0 as x goes to zero. This means that there must be an $\epsilon > 0$ such that $|g(x)| \geq \epsilon$ for arbitrarily large x's. Let $h \in \mathrm{B}(g, \frac{\epsilon}{2})$. Then $|g(x) - h(x)| < \frac{\epsilon}{2}$ for all x, and hence there must be arbitrarily large x's where $|h(x)| \geq \frac{\epsilon}{2}$ (the same x's where $|g(x)| \geq \epsilon$). Hence h does not converge to 0 as x goes to infinity, which means that $h \in X^c$. Thus for any $g \in X^c$, there is a ball $\mathrm{B}(g, \frac{\epsilon}{2})$ around g that also belongs to X^c, and hence X^c is open.

Alternative solution: Assume that $\{f_n\}$ is a Cauchy sequence in X; we must prove that it converges to an $f \in X$ in the uniform norm. First observe that for any $x \in [0, \infty)$, $|f_n(x) - f_m(x)| \leq \|f_n - f_m\|$, and hence $\{f_n(x)\}$ is a Cauchy sequence for every x. Since $\mathbb{R}$ is complete, $\{f_n(x)\}$ converges to a point which we call $f(x)$. We must prove that $\{f_n\}$ converges uniformly to f and that f belongs to X.

First observe that for a given ϵ, there is an $N \in \mathbb{N}$ such that $\|f_n - f_m\| < \frac{\epsilon}{2}$ for all $n, m \geq N$. This means that for any $x \in [0, \infty)$, $|f_n(x) - f_m(x)| < \frac{\epsilon}{2}$. Letting $m \to \infty$, we get $|f_n(x) - f(x)| \leq \frac{\epsilon}{2} < \epsilon$ for all $n \geq N$, and hence $\{f_n\}$

converges uniformly to f. As uniform convergence preserves continuity, f is continuous.

To prove that $f \in X$, it remains to show that $\lim_{x\to\infty} f(x) = 0$. Given $\epsilon > 0$, we must find a $K \in \mathbb{R}$ such that $|f(x)| < \epsilon$ for all $x \geq K$. Since $\{f_n\}$ converges uniformly to f, there is an $N \in \mathbb{N}$ such that $\|f - f_N\| < \frac{\epsilon}{2}$. As $f_N \in X$, there is a $K \in \mathbb{N}$ such that $|f_N(x)| < \frac{\epsilon}{2}$ for all $x \geq K$. This means that for $x \geq K$.

$$|f(x)| = |f(x) - f_N(x) + f_N(x)| \leq |f(x) - f_N(x)| + |f_N(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

This shows that $f \in X$ and completes the proof.