Ark6: Solutions

I have written down some very sketchy solutions to some of the exercises. I have only treated the ones given for the friday sessions, and it has been done rather hastly, so forgive me if there are errors. Still, I hope, they will be useful for you.

PROBLEM 1:

a) The terms of the sequence $\{a_n\} = \{(-1)^n\}$ oscillate between -1 and 1. Hence $\limsup a_n = 1$ and $\liminf a_n = -1$.

b) We have

$$\cos\frac{n\pi}{2} = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is divisibel by } 4 \\ -1 & \text{if } n \text{ is even, but not divisibel by } 4 \end{cases}$$

hence $\limsup a_n = 1$ and $\liminf a_n = -1$.

c) We have

$$\arctan n \sin\frac{n\pi}{2} = \begin{cases} 0 & \text{if } n \text{ is even} \\ \arctan n & \text{if } n \text{ is on the form } \ 4n+1 \\ -\arctan n & \text{if } n \text{ is on the form } \ 4n+3 \end{cases}$$

hence the subsequence $\{a_{4n+1}\}$ of $\{a_n\}$ tends to $\pi/2$, and the subsequence $\{a_{4n+3}\}$ to $-\pi/2$. Furthermore, $-\frac{\pi}{2} \leq \arctan x \leq \frac{\pi}{2}$ for all x; hence $\limsup a_n = \pi/2$ and $\liminf a_n = \pi/2$. ❑

PROBLEM 2: By the mean value theorem, there is a c between $n^2\pi^2$ and $t + n^2\pi^2$ such that

$$\cos\sqrt{t+n^2\pi^2} - \cos\sqrt{n^2\pi^2} = \frac{-\sin\sqrt{c+n^2\pi^2}}{2\sqrt{c+n^2\pi^2}}t,$$

and hence

$$\left|\cos\sqrt{t+n^2\pi^2} - \cos n\pi\right| \leq \frac{t}{2n\pi}.$$

Using that $\cos n\pi = (-1)^n$ we get

$$\left|\cos\sqrt{t+n^2\pi^2} - (-1)^n\right| \leq \frac{t}{2n\pi}.$$

This shows that the subsequence $\{a_{2n}\}$ converges to 1 and the subsequence $\{a_{2n+1}\}$ to -1; and of course, $\left|\cos\sqrt{t+n^2\pi^2}\right| \leq 1$ for all t. It follows that $\limsup a_n = 1$ and

$\liminf a_n = -1$. ❏

PROBLEM 3: We have

$$\frac{1/n-2}{1/n+3} \leq a_n = \frac{1+(-1)^n 2n}{1+3n} = \frac{1/n+2(-1)^n}{1/n+3} \leq \frac{1/n+2}{1/n+3}, \tag{$*$}$$

So $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ are to be found between $-\frac{2}{3}$ and $\frac{2}{3}$ — the limits of respectively the left and right side of $*$. The subsequence $\{a_{2n}\}$ tends to $\frac{2}{3}$, and the subsequence $\{a_{2n+1}\}$ tends to $-\frac{2}{3}$, as $n \to \infty$. Hence $\limsup_{n\to\infty} a_n = \frac{2}{3}$ and $\liminf_{n\to\infty} a_n = -\frac{2}{3}$ ❏

PROBLEM 7: There are several cases depending of the values of a and b.

1) $b = 0, a = 0$.Then $\frac{x+a}{x+b} = 1$. The function $\sin^2 \frac{1}{x}$ oscillates between 0 og 1 — it is always positive, takes the value 0 when $x = \frac{1}{n\pi}$ and the value 1 when $x = \frac{1}{(n+\frac{1}{2})\pi}$. Therefore $\liminf_{x\to 0} \frac{x+a}{x+b} \sin^2 \frac{1}{x} = 0$ og $\limsup_{x\to 0} \frac{x+a}{x+b} \sin^2 \frac{1}{x} = 1$.

2) $b = 0, a \neq 0$. If $a > 0$, then $\frac{x+a}{x+b} = 1 + \frac{a}{x}$ which tends to ∞ when x tends to zero through positive values of x, and to $-\infty$ when x goes to zero from below. Therefore $\liminf = -\infty$ og $\limsup = \infty$; The case $a < 0$ is treated in a similar way.

3) $b \neq 0$. Then $\lim_{x\to 0} \frac{x+a}{x+b} = \frac{a}{b}$. It follows that $\liminf = 0$, and $\limsup = \frac{a}{b}$ if $\frac{a}{b} > 0$, and that $\liminf = \frac{a}{b}$ and $\limsup = 0$ when $\frac{a}{b} < 0$. ❏

PROBLEM 8: We have $|\cos nx/(1+n^2)| \leq 1/(1+n^2)$. Since $\sum_{n=0}^{\infty} \frac{1}{1+n^2}$ converges, the series $\sum_{n=0}^{\infty} \frac{\cos nx}{1+n^2}$ converges uniformly by Weierstrass' M-test. ❏

PROBLEM 9:

a) We have $|a_n \cos nx| \leq |a_n|$. Now as $|a_n| \leq \frac{A}{n^s}$ for some positive constant A and $\sum_{n=0}^{\infty} \frac{1}{n^s}$ converges as $s > 1$, $\sum_{n=0}^{\infty} |a_n|$ converges by the comparison test. Hence $\sum_{n=0}^{\infty} a_n \cos nx$ converges uniformly by Weiestrass' M-test. As each function $a_n \cos nx$ is continuous, and the uniform limit of continuous functions is continuous, it follows that $\sum_{n=0}^{\infty} a_n \cos nx$ is continuous.

b) We have $|na_n \sin nx| \leq \frac{1}{n^{s-1}}$. The series $\sum_{n=0}^{\infty} \frac{1}{n^{s-1}}$ converges since $s-1 > 1$. Hence $-\sum_{n=0}^{\infty} na_n \sin nx$ converges uniformly. But this is the derived series of $\sum_{n=0}^{\infty} a_n \cos nx$, so it converges to the derivative of $f(x)$ by **proposition 4.2.5** on page 89 in Tom's notes.

c) This is just an inductive argumet using b). ❏

PROBLEM 12: If $r > 0$, the power series $\sum_{n=0}^{\infty} (\frac{x}{r})^n$ has radius of convergence r. Indeed,

$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \limsup_{n\to\infty} \frac{1}{r^n} = \frac{1}{r}$.

The power series $\sum_{n=0}^{\infty} n^n x^n$ has radius of convergence equal to 0 . Indeed, $\sqrt[n]{n^n} = n$ tends to ∞ as $n \to \infty$, and the radius of convergence equals

$$1/\limsup_{n\to\infty} \sqrt[n]{|a_n|} = 1/\limsup_{n\to\infty} \sqrt[n]{n^n} = 1/\limsup_{n\to\infty} n = 0.$$

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