Ark10: Exercises for MAT2400 — Fourier series and the outer measure

The exercises on this sheet cover the sections *4.9* to *5.1*. They are intended for the groups on Thursday, April 19 and Friday, April 20.

The distribution is the following: *Friday, April 20:* No 2, 7, 8, 9, 10, 11.
The rest for Thursday, April 19.

Key words: Fourier series, outer measure.

Fourier series

PROBLEM 1. Let $f(x)$ and $g(x)$ be two piecewise continuous functions both periodic with period 2π. We define the *convolution product* of f and g — which is denoted by $f \star g$ — by the formula

$$f \star g(x) = \int_{-\pi}^{\pi} f(x-t)g(t)\,dt. \qquad (❏)$$

a) Show that $f \star g$ is periodic with period 2π and that $f \star g = g \star f$.

b) Show that if $e_n(x) = e^{inx}$, then $e_n \star f = 2\pi c_n e_n$ where c_n is the n-th Fourier coefficient of f.

c) Show that if $T(x) = \sum_{k=-\nu}^{\mu} a_k e^{ikx}$ is a trigonometric polynomial, then $T \star f$ is also a trigonometric polynomial.

PROBLEM 2. Let $\chi(x)$ be the periodic function of period 2π which is given in $[-\pi, \pi]$ by

$$\chi(x) = \begin{cases} 0 & \text{if } -\pi \le x < -a \\ 1 & \text{if } -a \le x \le a \\ 0 & \text{if } \ a < x \le \pi \end{cases}$$

where a is a constant with $0 < a \le \pi/2$.

a) Sketch the graph of χ, and show that for any 2π-periodic, integrabel function g we have

$$\chi \star g(x) = \int_{x-a}^{x+a} f(u)\,du.$$

b) Show that $\chi \star \chi$ is given by

$$\chi \star \chi(x) = \begin{cases} 0 & \text{if } -\pi \leq x < -2a \\ x + 2a & \text{if } -2a \leq x \leq 0 \\ -x + 2a & \text{if } \ 0 < x \leq 2a \\ 0 & \text{if } \ 2a < x \leq \pi \end{cases}$$

c) Determine $(\chi \star \chi) \star \chi$.

PROBLEM 3. Define the Fejér kernel $F_N(x)$ by the folllowing formula where N is natural number:

$$F_N(x) = \frac{1}{N \sin x/2} \sum_{k=0}^{N-1} \sin(k + 1/2)x.$$

a) Show that the Fejér kernel satisfies

$$F_N(x) = \frac{\sin^2 \frac{Nx}{2}}{N \sin^2 \frac{x}{2}}.$$

HINT: The formula $2 \sin \alpha \sin \beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$ may be usuful with $\alpha = (k + 1/2)x$ and $\beta = x/2$.

b) Show that

$$F_N(x) = \sum_{k=0}^{N-1} D_k(x)$$

where $D_k(x)$ is the Dirichlet kernel.

c) Show that $F_N(0) = N$ and that $|F_N(x)| \leq \frac{\pi^2}{Nx^2}$ if $x \neq 0$.

d) Show that $\frac{1}{2\pi} \int_{-\pi}^{\pi} F_N(x)\, dx$=1.

e) Show that for any δ with $0 < \delta < \pi$ it holds true that $\int_{\delta}^{\pi} F_N(x)\, dx < \frac{\pi^2}{N}(\frac{1}{\delta} - \frac{1}{\pi})$.

PROBLEM 4. Let f a continuous, periodic function of period 2π, and let F_n be the Fejér kernel from problem 3.

a) Show that the sequence $\{G_n \star f\}$ converges uniformly to f.

b) Show that f can be uniformly approximated by trigonometrical polynomials.

PROBLEM 5. Let $f(x)$ be an integrabel, periodic function of period 2π.

a) Show that $\int_{-\pi}^{\pi} f(x)\, dx = \int_{-\pi}^{\pi} f(x + a)\, dx$ for any real number a.

b) Show that the Fourier coefficients of f satisfy $c_n = \frac{1}{4\pi} \int_{-\pi}^{\pi} (f(x) - f(x + \frac{\pi}{n}))e^{-inx}\, dx$.

c) Suppose now that the function $f(x)$ is what one calls Lipschitz continuous with Lipschitz constant α, where $\alpha \leq 1$. That means that there are constants A and δ such that

$$|f(x+h) - f(x)| \leq Ah^{\alpha}$$

for all x and for $|h| < \delta$. Show that then the Fourier constants c_n of $f(x)$ satisfy $|c_n| < \frac{A}{4\pi} n^{\alpha}$ for sufficiently big n.

d) Show that if the constant α in 5.c) is greater than one, then f is constant — so the condition $\alpha \leq 1$ in 5.c) has a reason.

The outer measure

PROBLEM 6. *(Tom's notes 5.1, Problem 1 (page 150)).* Show that the outer measure of any *countable* set is equal to zero.

PROBLEM 7. Show that if $A \subseteq R^d$ has outer measure zero and $B \subseteq A$, then $\mu^* B = 0$.

PROBLEM 8. *(Tom's notes 5.1, Problem 2 (page 150)).*

a) Show that the x-axis has outer measure zero in $\mathbb{R}^2$.

b) Show that a linear subspace $V \subseteq \mathbb{R}^{d+1}$ of dimension d has outer measure zero in $\mathbb{R}^{d+1}$. HINT: By choosing an appropriate orthonormal basis for $\mathbb{R}^{d+1}$ one may assume that $V = \{(x_1, \ldots, x_d, 0) : x_i \in \mathbb{R}\}$.

c) Can you draw a general conclusion from this?

PROBLEM 9. Let $A, B \subseteq \mathbb{R}$ be two intervals. Show that

$$\mu^* A = \mu^*(A \cap B) + \mu^*(A \cap B^c).$$

PROBLEM 10. *(Tom's notes 5.1, Problem 3 (page 150)).* Show that the outer measure μ^* on $\mathbb{R}$ is *translation invariant*; *i.e.*, for any $A \subseteq \mathbb{R}$ and any $x \in \mathbb{R}$ we have $\mu^*(A+x) = \mu^* A$ where $A + x = \{a + x : a \in A\}$.

PROBLEM 11. *(Basically Tom's notes 5.1, Problem 4 (page 150)).* Let $A \subseteq \mathbb{R}^d$ be any subset and let $x \in \mathbb{R}$ be a number. Denote by xA the set $xA = \{xa : a \in A\}$.

a) Show that $\mu^*(2A) = 2^d \mu^* A$.

b) What can you say about $\mu^*(3A)$?

c) Show that $\mu^*(-A) = \mu^* A$.

Versjon: Friday, March 30, 2012 11:17:09 AM