Ark11: Exercises for MAT2400 — Fourier series and Measurable sets

The exercises on this sheet cover the sections *5.1* to *5.3*. They are intended for the groups on Thursday, April 26 and Friday, April 27.

The distribution is the following: *Friday, April 27:* No 1, 2, 5, 6, 7, 9.
The rest for Thursday, April 26.

Key words: Fourier series, Measurable sets.

Measurable sets

Problem 1. *(Tom's notes 5.3, Problem 2 (page 160)).* If E_1 and E_2 are measurable subsets of $\mathbb{R}^d$, then

$$\mu(E_1) + \mu(E_2) = \mu(E_1 \cap E_2) + \mu(E_1 \cup E_2).$$

Problem 2. Let A and B be two sets. The *symmetric difference* between them is the set

$$A \triangle B = (A \setminus B) \cup (B \setminus A).$$

a) Show that if A and B are measurable, then $A \triangle B$ is measurable, and if A and B are of finite measure, then

$$\mu(A \triangle B) = \mu(A \cup B) - \mu(A \cap B).$$

b) Given a positive number a. Find examples of A and B with $\mu(A \cup B) = \mu(A \cap B) = \infty$ and

$$\mu(A \triangle B) = a.$$

Problem 3. *(Tom's notes 5.3, Problem 4 (page 160)).* Let $E \subseteq \mathbb{R}^d$ be a set of finite measure. Show that for any given $\epsilon > 0$, there is a compact subset $K \subseteq E$ such that $\mu(E \setminus K) < \epsilon$.

Problem 4. *(Tom's notes 5.3, Problem 3 (page 160)).* A subset of $\mathbb{R}^d$ is a called $\mathcal{G}_\delta$-set if it is the intersection of countably many open sets. It is called $\mathcal{F}_\sigma$-set if it is the union of countably many closed sets.

a) Explain why any $\mathcal{G}_d$-set and any $\mathcal{F}_\sigma$-set is measurable.

b) Show that if E is measurable, then there is a $\mathcal{G}_\delta$-set G with $\mu(E \triangle G) = 0$.

c) Show that if E is measurable, then there is a $\mathcal{F}_\sigma$-set F with $\mu(E \triangle F) = 0$.

PROBLEM 5. *(Tom's notes 5.3, Problem 5 (page 160)).* Assume that $\{E_n\}$ is a countable family of measurable subsets of $\mathbb{R}^d$ such that $\sum_{n=1}^{\infty} \mu(E_n) < \infty$. Show that the set

$$E = \{x \in \mathbb{R}^d : x \text{ belongs to infinitly many } \ E_n\}$$

is of measure zero.

PROBLEM 6. Show that the interval $[0, 1]$ is not countable by using the Lebesgue measure.

PROBLEM 7.

a) Show that if $\{E_n\}$ is a countable family of subsets of $\mathbb{R}^d$ of measure zero, then $\bigcup_{n=1}^{\infty} E_n$ is of measure zero.

b) Show that an arbitrary intersection of subsets of $\mathbb{R}^d$ of measure zero is of measure zero.

c) Is the set $\mathcal{N}$ of subsets of $\mathbb{R}^d$ of sets of measure zero a σ-algebra?

PROBLEM 8. In this exercise we will construct the *Cantor set* $\mathcal{C}$ which is a subset of $[0, 1]$ of measure zero *not* being enumerable.

We are going to recursively define a descending sequence of subsets C_n of $[0, 1]$, each one being the union of closed intervals, and whose intersection will be the Cantor set $\mathcal{C}$, *i.e.*, $\mathcal{C} = \bigcap_{n=1}^{\infty} C_n$.

We let $C_0 = [0, 1]$, and $C_1 = [0, 1/3] \cup [2/3, 1]$,*i.e.*, to get C_1 from C_0 we throw away the midle third of C_0. To get C_{n+1} from C_n we divide *each* of the closed intervals whose union equals C_n, in three equal parts and throw away the middel (open) interval.

a) Show by induction that C_n is the union of 2^n pairwise disjoint closed intervals each of length 3^{-n}.

b) Show that $\mu(C_n) = 2^n \cdot 3^{-n}$ and that $\mu(\mathcal{C}) = 0$.

c) Show by induction that the endpoints of the closed intervals that constitute C_n all are of the following form, where a_ν is either 0 or 2:

- $\sum_{\nu=1}^{n} a_\nu 3^{-\nu}$ if it is a *left* endpoint.
- $\sum_{\nu=1}^{n} a_\nu 3^{-\nu} + 3^{-n} = \sum_{\nu=1}^{n} a_\nu 3^{-\nu} + \sum_{\nu=n+1}^{\infty} 2 \cdot 3^{-\nu}$ if it is a *right* endpoint.

d) Show that $\mathcal{C}$ is not enumerable. HINT: $\mathcal{C}$ containes all numbers of the form $\sum_{\nu=1}^{\infty} a_\nu 3^{-\nu}$ where a_ν is either zero or two.

PROBLEM 9. A set $E \subseteq X$ where (X, d) is a metric space, is called nowhere dense in X if the complement of the closure of E is dense.

a) Show that E is nowhere dense in X if and only if the closure of E does not contain any nonempty open set.

b) Show that the Cantor set is nowhere dense in $[0, 1]$. HINT: Show that $\mathcal{C}$ does not contain any open nonempty interval.

PROBLEM 10. A metric space (X, d) is called *totally disconnected* if it has the property that for any pair of distinct points x and y from X, there exist two disjoint open sets U and V with $U \cup V = X$ such that $x \in U$ and $y \in V$.

a) Show that the sets U and V above are closed.

b) Show that the Cantor set is totally disconnected

FOURIER SERIES PROBLEM 11. The aim of this problem is to give example (due to our friend Fejér, by the way) of a *continuous* function in $[-\pi, \pi]$ whose Fourier series diverges at the origin.

a) Let n and m be two natural numbers. Show that:

$$\sum_{\nu=0}^{n-1} \frac{\cos(m+\nu)x}{n-\nu} - \sum_{\nu=1}^{n} \frac{\cos(m+n+\nu)x}{\nu} = \sin\left((m+n)x\right) \sum_{\nu=1}^{n} \frac{\sin \nu x}{\nu}.$$

b) Let $\{a_k\}$ be a sequence such that $3a_k < a_{k+1}$, and for each k put $m = n = a_k$ and let $Q_k(x) = \sin(2a_k x) \sum_{\nu=1}^{a_k} \frac{\sin \nu x}{\nu}$. Show that the series

$$\sum_{k=1}^{\infty} k^{-2} Q_k(x)$$

converges uniformly on $[-\pi, \pi]$ towards a continuous function $f(x)$. HINT: Use Weierstrass' M-test and the fact that $\sum_{\nu=1}^{n} \frac{\sin \nu x}{\nu}$ is bounded, see problem 4.c) on Ark9.

c) Show that the N-th term of the Fourier series of f is equal to

$$\begin{aligned} k^{-2}\frac{\cos(a_k+\nu)x}{a_k-\nu} \quad & \text{if } a_k \leq a_k+\nu = N \leq 2a_k - 1 \\ -k^{-2}\frac{\cos(2a_k+\nu)x}{\nu} \quad & \text{if } 2a_k+1 \leq 2a_k+\nu = N \leq 3a_k \\ 0 \quad & \text{for all other values of } N \end{aligned}$$

d) Let $s_r(x)$ be the r-th partial sum of the Fourier series of $f(x)$. Show that

$$|s_{2a_k}(0) - s_{a_k-1}(0)| > k^{-2} \log a_k$$

HINT: Use that $\sum_{\nu=1}^{n} \frac{1}{\nu} > \log n$.
e) Show that if we take $a_k = k^3$, then the Fourier series of $f(x)$ diverges at the origin.