Ark3: Exercises for MAT2400 — Spaces of continuous functions

The exercises on this sheet covers the sections *3.1* to *3.4* in Tom's notes. They are together with a few problems from Ark2 the topics for the groups on Thursday, February 16 and Friday, February 17. With the following distribution:

Thursday, February 16: No 3, 4, 5, 6, 9, 10, 12.
From Ark2: No 27, 28

Friday, February 17: No: 1,2, 7, 8, 11, 13

Key words: Uniform continuity, equicontinuity, uniform convergence, spaces of continuous functions.

Uniform continuity and equicontinuity

Problem 1. *(Tom's notes 1, Problem 3.1 (page 50)).* Show that x^2 is not uniformly continuous on $\mathbb{R}$. Hint: $x^2 - y^2 = (x+y)(x-y)$.

Problem 2. *(Tom's notes 2, Problem 3.1 (page 51)).* Show that $f\colon (0,1) \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$ is not uniformly continuous.

Problem 3.

a) Let I be an interval and assume that f is a function differentiable in I. Show that if the derivative f' is bounded in I, then f is uniformly continuous in I. Hint: Use the mean value theorem.

b) Show that the function $\sqrt{x}$ is uniformly continuous on $[1,\infty)$. Is it uniformly continuous on $[0,\infty)$?

Problem 4. Let $\mathscr{F} \subseteq C([a,b])$ denote a subset whose members all are differentiable on (a,b). Assume that there is a constant M such that $|f'(x)| \leq M$ for all $x \in (a,b)$ and all $f \in \mathcal{F}$. Show that $\mathscr{F}$ is equicontinuous. Hint: Use the mean value theorem.

Problem 5. Let $\mathscr{P}_n$ be the subset of $C([0,1])$ whose elements are the poly's of degree at most n. Let M be a positive constant and let $S \subseteq \mathscr{P}_n$ be the subset of poly's with coefficients bounded by M; *i.e.*, the set of poly's $\sum_{i=0}^{n} a_i T^i$ with $|a_i| \leq M$.

a) Show that S is equicontinuous.

b) Show that $\mathscr{P}_1$ is not equicontinuous.

Problem 6. Let λ be a constant with $0 < \lambda < 1$. Let $\mathscr{F} \subseteq C([0,\lambda])$ be the set

$$\mathscr{F} = \{x^s \mid s \in (1,\infty)\}.$$

Show that $\mathscr{F}$ is equicontinuous. HINT: Use problem 4. Show that $\phi(s) = s\lambda^{s-1}$ is a bounded function of s by finding the maximum value of $\log s + (s-1)\log\lambda$.

a) Is the corresponding set $\mathscr{F} \subseteq C([0,1])$ equicontinous?

Uniform convergence

PROBLEM 7. *(Basically Tom's notes 3.2, Problem 1 (page 54)).* Let $f_n\colon \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \frac{x}{n}$.

a) Show that $\{f_n\}$ converges pointwise, but not uniformly to 0.

b) Show that $\{f_n\}$ converges uniformly on every closed intervall $[a,b]$.

PROBLEM 8. *(Tom's notes 3.2, Problem 2 (page 54)).* Let $f_n\colon (0,1) \to \mathbb{R}$ be defined by $f_n(x) = x^n$. Show that f_n converges pointwise but not uniformly to 0.

PROBLEM 9. *(Basically Tom's notes 3, Problem 3.2 (page 54)).* Let $f_n\colon [0,\infty) \to \mathbb{R}$ be defined by $f_n(x) = e^{-x}(\frac{x}{n})^{ne}$.

a) Show that $f_n(x)$ converges pointwise. HINT: Use that $(\frac{x}{n})^n$ tends to 0 as n tends to ∞.

b) Find the maximun value of f_n. Does f_n converge uniformly? HINT: Work with $\log f_n(x)$ when you look for the max value.

c) If we let $g_n(x) = e^{-x}(\frac{x}{n})^{na}$ where $a > e$, show that then g_n converges uniformly.

PROBLEM 10. *(Tom's notes 3.2, Problem 4 (page 54)).* The function $f_n\colon (0,\infty) \to \mathbb{R}$ is defined by

$$f_n(x) = n(x^{\frac{1}{n}} - 1).$$

a) Show that $\{f_n\}$ converges pointwise to $\log x$. HINT: L'Hopitâl's rule can be useful.

b) Show that the convergence is uniform on any interval $(\frac{1}{N}, N)$ for $N \in \mathbb{N}$, but not on $(0,\infty)$.

The space $C(X,Y)$

PROBLEM 11. *(Tom's notes 3.3, Problem 1 (page 57)).* If f and g are given on $[0,1]$ by $f(x) = x$ and $g(x) = x^2$, find $\rho(f,g)$.

PROBLEM 12. Let a be a positive constant. Let $f(x) = x$ and $g(x) = \sin x$. Find $\rho(f,g)$ when we regard f and g as elements in $C([0,a])$.

PROBLEM 13. *(Basically Tom's notes 3.3, Problem 2 (page 57)).* Let $f(x) = \sin x$ and $g(x) = \cos x$ regarded as functions on the interval $[0, 2\pi]$. Determine $\rho(f, g)$. What is $\rho(f, g)$ if we regard f and g as functions on $[\pi, 2\pi]$?

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