Ark11: Solutions

I have written down some very sketchy solutions to some of the exercises. I have only treated the ones given for the friday sessions, and it has been done rather hastily, so forgive me if there are errors. Still, I hope, they will be useful for you.

PROBLEM 1: There is a disjoint decomposition:

$$E_1 \cup E_2 = (E_1 \setminus E_1 \cap E_2) \cup \; E_2.$$

Hence by additivity of μ we get:

$$\mu(E_1 \cup E_2) = \mu(E_1 \setminus E_1 \cap E_2) + \mu(E_2) = \mu(E_1) - \mu(E_1 \cap E_2) + \mu(E_2).$$

❑

PROBLEM 2:

a) We have $A \triangle B = A \cup B \setminus A \cap B$ since $A \triangle B$ consists of the points of A and B which are not contained in both sets. Hence $(A \triangle B) \cup (A \cap B) = A \cup B$, and as this is a disjoint decomposition, by additivity, we obtain since $\mu(A \cap B) < \infty$:

$$\mu(A \triangle B) = \mu(A \cup B) - \mu(A \cap B).$$

b) Take for example $A = (0, \infty)$ and $B = (a, \infty)$. ❑

PROBLEM 5: Clearly x belongs to infinitly many of the sets E_n if and only if x belongs to the sets $\bigcup_{n \geq k} E_k$ for *all* k. Hence

$$A = \{x : x \text{ belongs to infintly many } E_n\} = \bigcap_{k=1}^{\infty} \bigcup_{n \geq k} E_n.$$

As the sets in the intersection form a descending sequence, we have where we also use subadditivity:

$$\mu A = \lim_{k \to \infty} \mu(\bigcup_{n \geq k} E_n) \leq \lim_{k \to \infty} \sum_{n \geq k} \mu E_n = 0.$$

The last equality follows, since $\sum_{n=1}^{\infty} \mu E_n < \infty$. ❑

PROBLEM 6: If $[0, 1]$ were countable, it would be of measure zero, as this is true for all countable set. It is obviously not the case, as $\mu([0, 1]) = 1$. ❑

PROBLEM 9:

a) The subset E^c is dense if and only if it has a nonempty intersection with every open nonempty set. But this is equivalent to no nonempty open set being contained in E.

b) Recall that the Cantor set is the intersection $\bigcap_{n\geq 1} C_n$. For the definition of the sets C_n see exercise 8.

As any open set in $\mathbb{R}$ is a union of open intervals, it suffuces to show that the Cantor set does not contain any open interval. Assume that I is an open interval in $[0,1]$ and let a be its length. If n is such that $3^{-n} < a$, then I is *not* contained in any interval of length 3^{-n}. In particular it is not contained in C_n, since C_n is the union of such intervals. ❑

PROBLEM 10:

a) As $X = U \cup V$ and $U \cap V = \emptyset$, we have $U = V^c$ and $V = U^c$. This shows that both U and V are closed, since complements of open sets are closed.

b) We refere to problem 8 for notation about the Cantor set.

Assume that $x, y \in \mathcal{C}$ are two distinct points, and assume that $x < y$. Let a be the distance between them, and let n be such that $3^{-n} < a$. Then x, y can not both be contained in the same interval of length 3^{-n}. Hence they lie in different subintervals of C_n, and there is a $z \notin C_n$ with $x < z < y$. Then $U = \mathcal{C} \cap (-1, z)$ and $V = \mathcal{C} \cap (z, 2)$ are two disjoint (obviously) open subsets (since open subsets of $\mathbb{R}$ intersect $\mathcal{C}$ in open sets) of $\mathcal{C}$ whose union is equal to $\mathcal{C}$ (since $z \notin \mathcal{C}$) and are such that $x \in U$ and $y \in V$. ❑

Versjon: Tuesday, May 8, 2012 12:05:41 PM