Ark12: Exercises for MAT2400 — Measurable and simple functions

The exercises on this sheet cover the sections *5.4* and *5.5*. They are intended for the groups on Thursday, May 3 and Friday, May 4.

The distribution is the following: *Friday, May 4:* No 1, 2, 4, 5, 6, 9, 10.
The rest for Thursday, May 3.

Key words: Measurable functions, characteristic functions, simple functions and two of Littlewoods three principles.

Measurable functions

Problem 1. Let $f\colon \mathbb{R}^d \to \overline{\mathbb{R}}$ be a measurable function.

a) Show that the the sets $f^{-1}(\infty) = \{x : f(x) = \infty\}$ and $f^{-1}(-\infty) = \{x : f(x) = -\infty\}$ are measurable.

b) Show that for any $a \in \overline{\mathbb{R}}$, the *level set* (sometimes called *the fibre* of f over a) $f^{-1}(x) = \{x : f(x) = a\}$ is measurable.

Problem 2. Show that a function $f\colon \mathbb{R}^d \to \overline{\mathbb{R}}$ is measurable if and only if $f^{-1}(I)$ is measurable for any interval I.

Problem 3. *(Tom's notes 5.4, Problem 9 (page 165)).* Let f and g be two measurable functions on $\mathbb{R}^d$. We shall write $f \sim g$ if f and g are equal almost everywhere. Show that this is an equivalence relation. That is:

(i) $f \sim f$

(ii) $f \sim g$ if and only if $g \sim f$

(iii) If $f \sim g$ and $g \sim h$, then $f \sim h$.

Problem 4. Let f and g be two functions on $\mathbb{R}^d$, and assume that $f(x) = g(x)$ almost everywhere. Show that if f is measurable, then g is measurable.

Problem 5. Let $E \subseteq \mathbb{R}^d$ be a measurable set and let $f\colon E \to \overline{\mathbb{R}}$ be a function. Let $\tilde{f}\colon \mathbb{R}^d \to \overline{\mathbb{R}}$ be the function we obtain by extending f by zero; that is

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in E \\ 0 & \text{if not .} \end{cases}$$

Show that $\tilde{f}(x)$ is measurable if and only if $f^{-1}(I)$ is measurable whenever $I = \{x \in \overline{\mathbb{R}} : x < r\}$ for an $r \in \mathbb{R}$.

PROBLEM 6. *(Tom's notes 5.4, Problem 6 (page 165)).* Let $\{E_n\}$ be a family of pairwise disjoint and measurable subsets of $\mathbb{R}^d$. Assume that $\bigcup_{n=1}^{\infty} E_n = \mathbb{R}^d$, and let $\{f_n\}$ be a sequence of measurable functions on $\mathbb{R}^d$. Show that the function defined by

$$f(x) = f_n(x) \text{ if } x \in E_n,$$

is measurable.

PROBLEM 7. *(Tom's notes 5.4, Problem 12 (page 165)).* A sequence $\{f_n\}$ of measurable functions on $\mathbb{R}^d$ is said to *converge almost everywhere* to f if there is a subset A of $\mathbb{R}^d$ of measure zero such that $\{f_n(x)\}$ converges to $f(x)$ for all $x \notin A$. Show that f is measurable.

Simple functions

PROBLEM 8. Let $A \subseteq \mathbb{R}^d$ be a set. Recall that the *characteristic function* or the *indicator function* of A is given by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

a) Assume that A_1 and A_2 are two subsets of $\mathbb{R}^d$. Show that $\chi_{A_1 \cap A_2} = \chi_{A_1}\chi_{A_2}$, that $\chi_{A_1 \cup A_2} = \chi_{A_1} + \chi_{A_2} - \chi_{A_1}\chi_{A_2}$ and that $\chi_{A^c} = 1 - \chi_A$.
b) Let A_i, $i = 1, 2, 3$, be three subsets of $\mathbb{R}^d$. Make a drawing and show that

$$\chi_{A_1 \cup A_2 \cup A_3} = \sum_{1 \leq i \leq 3} \chi_{A_i} - \sum_{1 \leq i < j \leq 3} \chi_{A_i}\chi_{A_j} + \chi_{A_1}\chi_{A_2}\chi_{A_3}.$$

c) Let $A_1, \ldots, A_n$ be n subsets of $\mathbb{R}^d$. Show by induction that

$$\chi_{A_1 \cup \cdots \cup A_n} = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq n} \chi_{A_{j_1}} \cdot \cdots \cdot \chi_{A_{j_k}}.$$

This is often called the "inclusion-exclusion principle".

PROBLEM 9. If two simple functions are equal almost everywhere, show that their intergrals are equal.

PROBLEM 10. *(Basically Tom's notes 5.4, Problem 5 (page 165)).* Let $A \subseteq \mathbb{R}^d$ be a subset.

a) Show that the characteristic function χ_A of A is measurable if and only if A is measurable.

b) Recall that a *simple* function $f(x)$ on $\mathbb{R}^d$ is a function which can be written as

$$f(x) = \sum_{i=1}^{n} a_i \chi_{A_i} \tag{✢}$$

where $a_1, \ldots, a_n \in \mathbb{R}$ and $A_1, \ldots, A_n$ are measurable subsets of $\mathbb{R}^d$. Show that simple functions are measurable.

c) Give an example to show that expression for $f(x)$ in ✢ is not unique.

Two of Littlewood's three principles

PROBLEM 11. *(Littlewood's first principle).* Show that if $E \subseteq \mathbb{R}$ is a measurable set and $\epsilon > 0$ is a given number, them there is a finite union of pairwise disjoint open intervals $\bigcup_{k=1}^{n} I_k$ such that $\mu(E \setminus \bigcup_{k=1}^{n} I_k) < \epsilon$. HINT: Every open set in $\mathbb{R}$ is a countable union of pairwise disjoint open intervals (**Theorem 5.2.9** in Tom's). Then use **Proposition 5.3.5** on page 159 in Tom's.

The principle is: "A measurable subset of $\mathbb{R}$ is nearly a finite union of open intervals."

PROBLEM 12. Recall that a *step function* on $\mathbb{R}$ is a function that may be written as $g = \sum_{k=1}^{n} a_i \chi_{I_k}$ where $a_1, \ldots, a_n$ are real numbers, and where the sets I_k all are finite *intervals* (not merely measurable sets as is the case for simple function).

a) Given any simple function $f(x)$ on $\mathbb{R}$ and any $\epsilon > 0$, show that there is a *step* function g such that $|f(x) - g(x)| < \epsilon$ except on a set of measure less than ϵ. That is,

$$\mu(\{x : |f(x) - g(x)| \geq \epsilon\}) < \epsilon.$$

HINT: Treat first the case of a characteristic function. Use problem 11 above.

b) Show that given any step function $g(x)$, we may find a *continuous function* $h(x)$ such that $|g(x) - h(x)| < \epsilon$ except on a set of measure ϵ; that is:

$$\mu(\{x : |g(x) - h(x)| \geq \epsilon\}) < \epsilon.$$

HINT: Threat first the case of a characteristic function — in that case you should be able to draw $h(x)$.

Problem 13. Let f be a boundet, measurable function defined on $\mathbb{R}$. Let m and M be constants with $m \leq f(x) < M$ for all $x \in \mathbb{R}$. Show that there is a simple function ϕ such that $|f(x) - \phi(x)| < \epsilon$ for all x.

Hint: Divide $[m, M)$ into n pairwise disjoint intervals $I_k = [a_k, b_k)$ each of length less than ϵ. Let $E_k = \{x : f(x) \in I_k\}$, and let ϕ be a suitable linear combination of the characteristic functions χ_{E_k}.

Problem 14. *(Littlewood's second principle).* Show that for any measurable function f on an interval $[a, b]$ which is finite almost everywhere and any $\epsilon > 0$, there is a *continuous* function $g(x)$ suh that

$$\mu(\{x : |f(x) - g(x)| \geq \epsilon\}) < \epsilon.$$

The principle is: "A measurable function which is finite a.e. is nearly continuous."