Ark12: Solutions

I have written down some very sketchy solutions to some of the exercises. I have only treated the ones given for the friday sessions, and it has been done rather hastly, so forgive me if there are errors. Still, I hope, they will be useful for you.

Problem 1:

a) We have $f^{-1}(-\infty) = \bigcap_{n=1}^{\infty}\{x : f(x) < -n\}$ which is measurable since countable intersections of measurable sets are measurable and since, by definition of f being measurable, the sets $\{x : f(x) < r\}$ are measurable for all real r. In a similar way, $f^{-1}(\infty) = \bigcap_{n=1}^{\infty}\{x : f(x) \geq n\}$, and the sets $\{x : f(x) \geq n\}$ are all measurable since they are the complements of the sets $\{x : f(x) < n\}$, and complements of measurable sets are measurable.

b) We just remark that the fibre $f^{-1}(a)$ is given by

$$f^{-1}(a) = \{x : x \geq a\} \cap \bigcap_{n=1}^{\infty}\{x : f(x) < a + \frac{1}{n}\},$$

where all the sets appearing are measurable since f is measurable. ❑

Problem 2: We first treat the case $I = (-\infty, r]$, and write

$$f^{-1}(I) = \{x : f(x) \leq r\} = \bigcap_{n=1}^{\infty}\{x : f(x) < r + \frac{1}{n}\}.$$

This shows that $f^{-1}(I)$ is measurable, countable intersections of measurables are measurable. Similarily, if $I = (r, \infty)$, we conclude by the equality

$$f^{-1}(I) = \{x : f(x) > r\} = \bigcap_{n=1}^{\infty}\{x : f(x) \geq r + \frac{1}{n}\},$$

where the sets are measurable as the complements of the measurable sets $\{x : f(x) < r + \frac{1}{n}\}$. We now know that inverse images of infinite intervals by a measurable function are measurable.

Finally, if I is a finite interval, we may write $f^{-1}(I)$ as the intersection of inverse images of infinite intervals. For example, if $I = [a, b)$, we have

$$f^{-1}([a, b)) = f^{-1}((-\infty, b)) \cap f^{-1}([a, \infty)).$$

❑

PROBLEM 4: Let $E = \{x : f(x) \neq g(x)\}$. By hypo this set is of measure zero, hence any subset of E is measurable. Let r be a real number. We have

$$\{x : g(x) < r\} = (E^c \cap \{x : f(x) < r\}) \cup (E \cap \{x : g(x) < r\}),$$

where all involved sets are measuarble, E^c since E is measurable, $\{x : f(x) < r\}$ since f is measurable and $E \cap \{x : g(x) < r\}$ since it is a subset of the zero-measure-set E. We conclude that the set $\{x : g(x) < r\}$, being expressed by intersections and unions of measurabel sets, is measurable. ❑

PROBLEM 5: Let r be a real number. If $r > 0$ we have

$$\{x : \tilde{f}(x) < r\} = \{x \in E : f(x) < r\} \cup E^c,$$

where the two sets both are measurable. If $r \leq 0$, then

$$\{x : \tilde{f}(x) < r\} = \{x : f(x) < r\}$$

which by hypo is measurable. ❑

PROBLEM 6: Let r be a real number. We find

$$\{x : f(x) < r\} = \bigcup_{n=1}^{\infty} \{x \in E_n : f(x) < r\} =$$

$$= \bigcup_{n=1}^{\infty} \{x \in E_n : f_n(x) < r\} = \bigcup_{n=1}^{\infty} E_n \cap \{x : f_n(x) < r\}.$$

❑

PROBLEM 9: It will be sufficient to show that the intergral of a simple function f which is zero almost everywhere, is zero (Apply that result to the difference of the two functions in the problem). Let $f = \sum_{n=1}^{m} a_n \chi_{E_n}$ be the standard form of f where all $a_n \neq 0$. By hypo, each $\mu(E_n) = 0$, and therefore

$$\int f \, d\mu = \sum_{n=1}^{m} a_n \mu(E_n) = 0.$$

❑

PROBLEM 10:

a) Assume that A is measurable and let r be a real number. Then $\{x : \chi_A(x) < r\} = A^c$ when $r < 1$ and $\{x : \chi_A(x) < r\} = \mathbb{R}^d$ when $r \geq 1$. In both cases it is measurable.

Assume that χ_A is measurable. We have $A = \{x : \chi_A(x) \geq 1\}$ which we know is measurable by problem 1.

b) Each of the functions $a_i\chi_{A_i}$ is measurable by a) and the fact the a constant times a measurabel function is measurable. We also know that sums of measurable functions are measurable. And that does it.

c) Let $E_1 = (0, 1)$ and $E_2 = (0, 2)$ and $f = \chi_{E_1} + \chi_{E_2}$. Then we also have

$$f = 2\chi_{E_1} + \chi_{E_3}$$

where $E_3 = [1, 2)$. ❏