## Ark1: Solutions

I have written down some very sketchy solutions to some of the exercises. I have only treated the ones given for the friday sessions, and it has been done rather hastly, so forgive me if there are errors. Still, I hope, they will be useful for you.

PROBLEM 1: In all the three cases it is not too difficult to see that the two first axioms — positivity and symmetry — are satisfied. We check the triangle inequality:

*The Manhattan metric*, $d_{MN}$: We use the good old triangle inequality:

$$d_{MN}(x,y) = \sum_i |x_i - y_i| = \sum_i |x_i - z_i + z_i - y_i| \leq \sum_i |x_i - z_i| + \sum_i |z_i - y_i| = \\ = d_{MH}(x,z) + d_{MN}(z,y).$$

*The sup-norm metric:*

$$\begin{aligned} d(x,y) &= \sup\{|x_i - y_i| : 1 \leq i \leq n\} = \sup\{|x_i - z_i + z_i - y_i| : 1 \leq i \leq n\} \\ &\leq \sup\{|x_i - z_i| + |z_i - y_i| : 1 \leq i \leq n\} \\ &\leq \sup\{|x_i - z_i| : 1 \leq i \leq n\} + \sup\{|z_i - y_i| : 1 \leq i \leq n\} \\ &= d(x,z) + d(z,y). \end{aligned}$$

*The Hamming metric:* Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ and $c_1, \ldots, c_n$ be three secret messages. Recall that the Hamming distance between two messages is the number of places where the two differ. We want to show that

$$d(a,b) \leq d(a,c) + d(c,b). \tag{$\star$}$$

Only places where $a$ and $c$ differ contributes to the left side of $(\star)$, so we solely have to show that such a place also contributes to the right side. But $c_i$ can not be equal to *both* $a_i$ and $b_i$ when those two are different. Hence there is a contribution either to $d(a,c)$ or $d(c,b)$ (or to both).

□

PROBLEM 2: This is done in Tome's notes: **Proposition 2.1.2** on page 23. □

PROBLEM 3: This is basically the same argument as in the part of problem 1 about the sup-norm metric: We get it by taking sup of the inequality

$$|f(x) - g(x)| \leq |f(x) - h(x)| + |h(x) - g(x)| \leq d(f,h) + d(h,g).$$

$\square$

PROBLEM 6:

a) $z$ must lie between $x$ and $y$.

b) The point $z$ must lie on the line segment connecting $x$ and $y$.

c) The point $z$ must lie in the block having $x$ and $y$ as corners.

$\square$

PROBLEM 8:

a) In this case the triangel inequality follows from the old one:

$$|f(x) - f(y)| \leq |f(x) - f(z)| + |f(z) - f(y)|,$$

and this is true for all functions $f$. Symmetry is obvious. To get positivity, or more precisely that $d(x, y) = 0$ only for $x = y$, we must have that $f(x) = f(y)$ implies $x = y$; that is, $f$ is injective.

b) The function $\frac{1}{x}$ is injective; use a). $\square$

PROBLEM : Assume there is a positiv constant $M$ such that $d_1(x, y) \leq Md_2(x, y)$ for all points $x, y \in X$.

Let $\{x_n\}$ be a sequence converging to $x$ in the metric $d_2$. We shall show that $\{x_n\}$ also converges to $x$ with respect to $d_1$. Let $\epsilon > 0$ be given. There exists an $N$ such that $d_2(x_n, x) < \epsilon/M$ when $n > N$. Then $d_1(x_n, x) < Md_2(x_n, x) < M\epsilon/M = \epsilon$ whenever $n > N$. Hence $\{x_n\}$ converges to $x$ also in the $d_1$ metric. The converse follows by switching the roles of $d_1$ and $d_2$.

$\square$

PROBLEM 11: Let $X$ be a finite set with two metrics $d_1$ and $d_2$. Let $M = \sup\{d_1(x, y)/d_2(x, y) : x, y \in X \text{ and } x \neq y\}$. This is finite since we take the supremum of a finite set. Clearly $d_1(x, y) \leq Md_2(x, y)$ for all $x, y \in X$. Switching the roles of the two metrics, we see that they are equivalent. $\square$

PROBLEM I: n the Manhattan metric, the ball is the square with corners $(\pm 1, 0)$ and $(0, \pm 1)$.

In the sup-norm metric, the ball is the square with corners $(\pm 1, \pm 1)$.

In the standard metric on $\mathbb{R}^2$, it is the circle with centre in the origin and radius one. $\square$

PROBLEM 14: It will be enough to show that if $d_2(x,y) \leq Md_1(x,y)$, then every open set in the $d_2$ metric is open in $d_1$. ( By switching roles of the metrics we see that the two metrics have the same open sets. Closed sets being the complements of open ones, it follows that they also have the same closed sets.) So let $U$ be open in $d_2$, and let $x \in U$. By definition there is then a ball $B_{d_2}(x;r) \subseteq U$. But $B_{d_1}(x;r/M) \subseteq B_{d_2}(x;r) \subseteq U$, so every point in $U$ has a $d_1$-ball round it contained in $U$; hence $U$ is open in $d_1$-metric. □

PROBLEM 15: Let us show that the complememnt $X^c = X \setminus \{x\}$ is open. Let $y \in X^c$, *i.e.*, $y \neq x$, and let $r = d(x,y)/2$. Then the ball $B = B_d(y;r)$ does not contain $x$, hence $B \subseteq X^c$. This shows that $X^c$ is open, and therefore $X$ is closed. A finite union of closed sets is closed, so any finite set is closed; in particular if it has two elements. □

PROBLEM 16:

a) $A$ is closed if $t = -1$; neither closed nor open in the two other cases.

b) The interior points of $A$ is the set $\{(x,y) \in \mathbb{R}^2 : y^2 < x \text{ and } x > t\}$

The exterior points of $A$ is the set $\{(x,y) \in \mathbb{R}^2 : y^2 > x \text{ or } x < t\}$

If $t \geq 0$, the boundary is the set

$$\{(x,y) \in \mathbb{R}^2 : y^2 = x \text{ and } t > x\} \cup \{(x,y) \in \mathbb{R}^2 : x = t \text{ and } y^2 \leq x\}.$$

If $t < 0$, then the boundary is the parabola $\{(x,y) \in \mathbb{R}^2 : y^2 = x\}$.

c) In all three caes the closure of $A$ can be described as the set $\{(x,y) \in \mathbb{R}^2 : y^2 \leq x \text{ and } x \geq t\}$ □

PROBLEM 19: Let $U = \{f(x) : \int_a^b f(x)\,dx > 1\}$, and let $f \in U$. Let $A = \int_a^b f(x)\,dx$ and let $r = (A-1)/2$. Then if $g(x) \in B(f;r)$, we have $\int_a^b (g(x) - f(x)) > -r$, and thus

$$\int_a^b g(x)\,dx = \int_a^b (g(x) - f(x))\,dx + \int_a^b f(x)\,dx \geq A - r = 1 + A/2 > 1.$$

Hence the ball $B(f;r) \subseteq U$, and $U$ is open. □

PROBLEM 23:

a) One easily checks that for real numbers $d$ and $r$ with $r < 1$, there is an equivalence

$$\frac{d}{1+d} < r \Leftrightarrow d < \frac{r}{1-r}.$$

This shows that $B_e(a;r) = B_d(a;\frac{r}{1-r})$ for $0 < r < 1$.
b) Take for example $X = \mathbb{R}$ and the usual Euclidian metric $d(x,y) = |x-y|$. The metric $d$ is not bounded, *i.e.,* we can make $d(x,y)$ as big as we want by chosing $x$ and $y$ far apart. However $e(x,y)$ is always less than one. This shows that two metrics can not be equivalent ( there is no number $M$ such that $d(x,y) \leq Me(x,y)$ for all $x$ and $y$). $\square$

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