Ark2: Solutions

I have written down some very sketchy solutions to some of the exercises. I have only treated the ones given for the friday sessions, and it has been done rather hastly, so forgive me if there are errors. Still, I hope, they will be useful for you.

PROBLEM 1:

a) The sequence $\{n\}$ is Cauchy in the metric d: We compute:

$$\left|\frac{1}{a_n} - \frac{1}{a_{n+k}}\right| = \left|\frac{1}{n} - \frac{1}{n+k}\right| = \frac{k}{n(n+k)} = \frac{1}{n(1+n/k)} < \frac{1}{n}.$$

Given $\epsilon > 0$; if $n > 1/\epsilon$, then $d(a_n, a_{n+k}) < \epsilon$ for all $k \in \mathbb{N}$.

It is not convergent: Suppose it were, and let $a \in \mathbb{R}^+$ be the limit. We compute:

$$d(n,a) = \left|\frac{1}{n} - \frac{1}{a}\right| = \frac{|a-n|}{na} = \frac{|a/n-1|}{a} > \frac{2}{a}$$

for n big, *i.e.*, for $n > 2a$.

b) No, $\{\frac{1}{n}\}$ is not Cauchy: We have $d(\frac{1}{n}, \frac{1}{n+k}) = |n-(n+k)| = k$, which can be made bigger than most ϵ-s.

c) Assume that $\{a_n\}$ converges to $a \in \mathbb{R}^+$ in the standard metric. We compute:

$$d(a_n, a) = \left|\frac{1}{a_n} - \frac{1}{a}\right| = \frac{|a-a_n|}{aa_n}. \tag{✤}$$

Now, let $\epsilon > 0$ be given. Since $a > 0$, the inequality $|a - a_n| < a/2$ holds true for n sufficently big. Hence $a_n > a/2$. For n perhaps even bigger, $|a - a_n| < (a^2/2)\epsilon$. From this we obtain using (✤):

$$d(a_n, a) < \frac{|a-a_n|}{aa_n} < \frac{(a^2/2)\epsilon}{a(a/2)} = \epsilon.$$

We have shown that $\{a_n\}$ converges to a in the metric d.

The other way around: Assume that $\{a_n\}$ converges to $a \in \mathbb{R}$ in the metric d. Then for n sufficiently big the following is true $\left|\frac{1}{a_n} - \frac{1}{a}\right| < \frac{1}{2a}$, from which it follows that $a_n < 2a$. By (✤) we then get:

$$|a - a_n| = aa_n\left|\frac{1}{a_n} - \frac{1}{a}\right| = aa_n d(a_n, a) < 2a^2 d(a_n, a). \tag{✢}$$

So, if $\epsilon > 0$ is given, we may, since $\{a_n\}$ converges in the metric d, find an N such that for $n > N$ we have $d(a_n, a) < \epsilon/2a^2$. But then $|a - a_n| < \epsilon$ by (✢). Hence $\{a_n\}$ converges to a in the usual metric. □

PROBLEM 5: Let $I \subseteq [0, 2\pi]$ be an open interval. If $\pi/2 \in I$ and $3\pi/2 \in I$, then $f(I) = [-1, 1]$ (by the intermediat value theorem and the facts that $\sin \pi/2 = 1$ and $\sin 3\pi/2 = -1$.) Assume that $\pi/2 \notin I$. Then $\sin x$ has no critical point[1] in I, hence has no maximal value in I (I is open). Therefore $f(I)$ is not closed (it has no "top point").

If $3\pi/2 \notin I$, an analoguous agument, with maximum replaced by minimum, shows that $f(I)$ is not closed (having no "bottom point"). So, the conclusion is that $f(I)$ is closed if and only if both $\pi/2$ and $3\pi/2$ are in I.

$f(I)$ is open if and only if f does neither have a maximum point nor a minimum point in I, which means that $I \subseteq (\pi/2, 3\pi/2)$. □

PROBLEM 6:

a) We have for two functions $f, g \in C([0, 1])$:

$$|E_a(f) - E_a(g)| = |f(a) - g(a)| \leq \sup\{|f(x) - g(x)| : x \in [0, 1]\} = \rho(f, g),$$

where, as usual, ρ denotes the sup-norm metric . Then it is clear that E_a is continuous: You may take $\delta = \epsilon$.

b) No, E_a is not continuous in the integral metric. Any sequence of functions $\{f_n\}$ with $f_n(a) = 1$, but with the area betsween the graph of f_n and the x-axis tending to zero when n tends to infinity, will give an example: Then $1 = E_a(f) \neq E_a(0) = 0$, but we can find elements from the sequence arbitrarly near 0 in the integral metric. If you want an explicit example, the following sequence of functions will do (make a drawing and compute the areas):

$$f(x) = \begin{cases} 0 & \text{if} \quad 0 \leq x \leq \frac{1}{2} - \frac{1}{n} \\ n(x - (\frac{1}{2} - \frac{1}{n})) & \text{if} \quad \frac{1}{2} - \frac{1}{n} < x \leq \frac{1}{2} \\ -n(x - (\frac{1}{2} + \frac{1}{n})) & \text{if} \quad \frac{1}{2} < x \leq \frac{1}{2} + \frac{1}{n} \\ 0 & \text{if} \quad \frac{1}{2} + \frac{1}{n} \leq x \leq 1. \end{cases}$$

□

PROBLEM 9: Clearly $f_n(x) = \frac{1}{n} \sin nx$ tends to zero in the sup-norm metric, indeed $\left|\frac{1}{n} \sin nx\right| < \frac{1}{n}$. We have $f_n'(x) = \cos nx$, and this sequence does not tend uniformly to

[1] Recall that a critical point is a point where the derivativ vanishes

zero (not even pointwise), since for any n , $\cos(n \cdot 0) = \cos 0 = 1$. Thus the map D is not continuous at 0. $\square$

Problem 11:

a) The set $\{(x, y) : x \geq 0, y \geq 0\}$ is complete, being closed in the complete space $\mathbb{R}^2$.

b) The set $A = \{(x, y) : x^2 + y^2 > 0\}$ is not complete; any sequence in $\mathbb{R}^2$ converging to the origin is a Cauchy in sequence A not converging in A.

c) The set $B = \mathbb{R}^2 \setminus \mathbb{Q} \times \mathbb{Q}$ is not complete. For example is the sequence $(\frac{\sqrt{2}}{n}, \frac{\sqrt{2}}{n})$ a Cauchy sequence from B (since $\sqrt{2}$ is not rational) which does not converge in B (since $(0, 0) \notin B$). $\square$

Problem 12: NB: New version of this exercise.

If $\{x_n\}$ is a Cauchy sequence from $A \cap B$, it converges to an element belonging both to A and B both being complete.

Assume that $\{a_n\}$ is a Cauchy sequence from the union $A \cup B$. Let $\{a_{n_i}\}$ be the subsequence of $\{a_n\}$ whose terms are the terms of $\{a_n\}$ belonging to A, and let $\{a_{m_i}\}$ be the one of all the terms in $\{a_n\}$ being elements in B.

Now, the sequence $\{a_{n_i}\}$ is Cauchy, being a subsequence of a Cauchy sequence, hence it converges to $a \in A$ since A is complete. In a similar way $\{a_{m_i}\}$ converges to an element b in B since B is complete.

As $d(a_{n_i}, a_{m_i})$ tends to zero when i tends to infinity, $d(a, b) = 0$, and therefore $a = b$. Hence $\{a_n\}$ converges to a (which is equal to b), and $A \cup B$ is complete. $\square$

Problem 15:

a) Each of the conditions $0 \leq x_i$ defines a closed set (they define sets being inverse images of closed intervals by continuous maps). Likewise, the condition $\sum_{i=1}^{n} x_i = 1$ also defines a closed set. The intersection of closed sets being closed, it follows that Δ_n is closed. It is also bounded since for each j:

$$0 \leq x_j \leq \sum_{i=1}^{n} x_i = 1.$$

Hence Δ_n is compact since closed and bounded sets in $\mathbb{R}^n$ are compact (by the Heine-Borel theorem).

b) In this case, the set is not compact. It is not bounded, since x_5 can be as big as we want. $\square$

PROBLEM 23: We have:

$$|f(x) - f(y)| = \left|x^2 - y^2\right| = |(x-y)|\,|(x+y)| \leq \frac{2}{3}\,|x-y|\,,$$

since both $|x|$ and $|y|$ are less than $1/3$. Hence f is a contraction. The fixed point satisfies the second degree equation

$$x = x^2 + 1/9,$$

which has the solution $\frac{3-\sqrt{5}}{6}$ in $[-1/3, 1/3]$. $\square$

PROBLEM 24:
a) We find:

$$|F(x,y) - F(x',y')| = \left|(\frac{1}{3}(x-x'), \frac{1}{5}(y-y'))\right| \leq \frac{1}{3}\,|(x,y)-(x',y')|\,,$$

where $|v|$ denotes the usual length of a vector.
b) The fixed point satisfies the equation: $(x,y) = (\frac{1}{3}x+1, \frac{1}{5}y+1)$ which has the solution $x = \frac{3}{2}$ and $y = \frac{5}{4}$. $\square$