Ark4: Solutions

I have written down some very sketchy solutions to some of the exercises. I have only treated the ones given for the friday sessions, and it has been done rather hastly, so forgive me if there are errors. Still, I hope, they will be useful for you.

PROBLEM 1: The differential equation is separable:

$$\frac{y'}{1+y^2} = 1,$$

and it gives $\arctan y = x + C$ by integration; hence $y = \tan(x + C)$. If $\tan(0 + C) = 0$, then $C = 0$ and the solution is equal to $\tan x$; a solution which is valid on $(-\pi, \pi)$. As the solution tends to ∞ at π it can not be extended beyond π.

$1 + y^2$ is not uniformly Lipschitz since

$$1 + y^2 - (1 + z^2) = y^2 - z^2 = (y - z)(y + z),$$

and if $\epsilon > 0$ and $\delta > 0$ are given, we can by setting $z = y + \delta$ and choosing $y > \epsilon/2\delta$, get

$$\left|1 + y^2 - (1 + z^2)\right| = |(y - z)(y + z)| = \delta(2y + \delta) \geq \delta 2y \geq \epsilon.$$

□

PROBLEM 2:

a) We have that $y'(t) = \frac{3}{2}(t-a)^{\frac{1}{2}} = \frac{3}{2}y^{\frac{1}{3}}$ if $t > a$. For $t < a$ both y' and y are identical zero, and they satisfy obviuously $y' = \frac{2}{3}y^{\frac{1}{3}}$.

$y(t)$ is differentiable for $t = a$: The differential quotient from the right is

$$\frac{(a+h-a)^{\frac{3}{2}}}{h} = \frac{h^{\frac{3}{2}}}{h} = h^{\frac{1}{2}} \to 0$$

when $h \to 0$. Clearly the differential quotient from the left is identically zero, so the to limits are equal, and y is differentiable at $t = 0$ with derivative equal 0; and this fits in the differential equation.

b) The mean value theprem gives us

$$y^{\frac{1}{3}} - z^{\frac{1}{3}} = \frac{1}{3}c^{-\frac{2}{3}}(y - z)$$

where c is some number between y and z. Given any $\epsilon > 0$ and $\delta > 0$, let $y - z = \delta$, then

$$y^{\frac{1}{3}} - z^{\frac{1}{3}} = \frac{1}{3}c^{-\frac{2}{3}}(y-z) = \frac{1}{3}c^{-\frac{2}{3}}\delta \geq \frac{1}{3}y^{-\frac{2}{3}}\delta$$

since $c < y$. Now choose $y > (\frac{1}{3}\frac{\delta}{\epsilon})^{\frac{3}{2}}$. That gives $\left|y^{\frac{1}{3}} - z^{\frac{1}{3}}\right| > \epsilon$. □

PROBLEM 3:
a) If $(b_{ij}) = Ay$, then $b_{ij} = \sum_{j=1}^{n} a_{ij}(t)y_j$. Hence

$$|b_{ij}| = \left|\sum_{j=1}^{n} a_{ij}(t)y_j\right| \leq \sum_{j=1}^{n} |a_{ij}(t)|\,|y_j| \leq M\,\|y\|\,,$$

since $M = \sup\{|a_{ij}(t)| : t \in [a,b] \text{ and } 1 \leq i,j \leq n\}$ and $\|y\| = \sup\{|y_j| : j = 1,\dots,n\}$.

If y and z are two members of $\mathbb{R}^n$, we have by linearity of A and by what we just did:

$$\|Ay - Az\| = \|A(y-z)\| \leq nM\,\|y - z\|\,,$$

so A is uniformly Lipschitz on $[a,b] \times \mathbb{R}^n$ with constant nM.
b) We checked that Ay is uniformly Lipschitz on $[a,b] \times \mathbb{R}^n$ so by theorem 3.4.2 from Tom's, we conclude that the initaial value problem

$$y'(t) = Ay(t) \qquad y(0) = y_0$$

has a unique solution for $t \in [a,b]$.

□

PROBLEM 4: Since $y(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$ we get $y' = \begin{pmatrix} u' \\ v' \end{pmatrix}$ so if y is a solution of the initial value problem

$$y'(t) = f(t, y(t)) \qquad y(0) = \begin{pmatrix} a \\ b \end{pmatrix},$$

we get from the definition of f

$$f(t,u,v) = \begin{pmatrix} v \\ g(t,v,u) \end{pmatrix}$$

the following: $v = u'$ and $v' = g(t,v,u) = g(t,u',u)$, *i.e.*, $u'' = g(t,u',u)$, and that is what we want. The initial condition $y(0) = \begin{pmatrix} a \\ b \end{pmatrix}$ translates into $u(0) = a$ and $u'(0) = v(0) = b$. □

PROBLEM 5: One way of doing this is to look at points in $\mathbb{R}^n$ whose coordinates are of the form $k/10^n$ where $n \in \mathbb{N}$ and $k \in \mathbb{Z}$. They are clearly form a countable set A (*i.e.,* , as subset of the rationals which is countable). Given a point $x \in \mathbb{R}^n$, then the decimal expansion of the coordinates of x give a sequence in A converging to x. □

PROBLEM 6: Assume that A is dense. Let $x \in X$ and let a_n be a sequence from a converging to x. If $r > 0$, there is an N such that for $n > N$ then $d(x, a_n) < r$, *i.e.,* $a_n \in B(r; x)$.

The other way around: Let $x \in X$. For every $n \in \mathbb{N}$ there is at least one element from A in $B(r; a)$. Pick one, and call it a_n. Then — almost by definition — $\{a_n\}$ is a sequence from A which converges to x. □

PROBLEM 7: Any real number has an expantion as a binary fraction: $x = a_1 a_2 \ldots a_r, b_1 b_2, \ldots$. where all a_i's and b_i's are $0, 1$ or 2. Hence there is a sequence from $A = \{k/2^n : n \in \mathbb{N}, k \in \mathbb{Z}\}$.

Or a more formal proof: Fix $x \in \mathbb{R}$. Let $s = \sup\{y \in A : y \leq x\}$. We claim that $s = x$. That will do, since it certainly implies there is a sequence from A converging to x. If $s < x$, let n be such that $2^{-n} < x - s$ and pick and element a from A such that $s - a < 2^{-n}$. Then $a + 2^{-n} \in A$, but $s < a + 2^{-n} < x$, which is a contradiction. □

PROBLEM 8:

a) *The sequence $\{f_n(x)\}$ is equicontinuous:* Indeed, if $\epsilon > 0$ is given, we have

$$d(f_n(x), f_n(y)) \leq \sigma_n d(x, y) < d(x, y)$$

where $\sigma_n < 1$ is the contraction factor of f_n. We therefore let $\delta = \epsilon$. This clearly gives $d(f_n(x), f_n(y))\epsilon$ whenever $d(x, y) < \delta$, and hence $\{f_n(x)\}$ is equicontinuous.

The sequence is bounded: Let the diameter of K be denoted by κ, *i.e.,* $\kappa = \sup\{d(x, y) : x, y \in K\}$, which is finite since K is compact. Clearly it holds true that $d(f_n(x), f_n(y)) < \kappa$, and it follows that $\{f_n(x)\}$ is a bounded sequence.

It follows by A&A — or more precisely by theorem 3.5.5 in Tom's — that our sequence $\{f_n(x)\}$, being bounded and equicontinuous, has a convergent subsequence.

b) By renaming (*i.e.,* replacing the orignal sequence by the uniformly convergent subsequence) we may assume that the sdequence $\{f_n(x)\}$ is uniformly continuous.

Each f_n is a contraction and has therefore a fixed point after Banach's Fixed point theorem (K being compact is complete). Let x_n be that point. Then sequence $\{x_n\}$ has a convergent subsequence, say $\{x_{n_i}\}$, since K is compact, and we let x be the limit of that sequence. Then — after lemma 3.5.7 in Tom's — we get

$$x = \lim_{i\to\infty} x_{n_i} = \lim_{i\to\infty} f_{n_i}(x_{n_i}) = f(\lim_{i\to\infty} x_{n_i}) = f(x),$$

and x is a fixed point for f.
c) Obviously $f_n(x) = (1-1/n)x$ is a contraction with factor $\sigma_n = (1-1/n)$. In the limit we get $f(x) = x$, and the convergence is uniform since $\sup\{f_n(x) : x \in [0,1]\} = 1-1/n$ is convergent. The limit $f(x) = x$ is not a contraction (the reason being that σ_n tends to one). □

PROBLEM 9. 9We intend ti use the A&A theorem, and must check that $\mathscr{K}$ is closed, bounded and equicontinuous:
$\mathscr{K}$ is bounded: By the assumptions we have

$$|f(x)| = |f(x) - f(0)| \leq K\,|x| \leq K.$$

for all $f \in \mathscr{K}$ and all $x \in [-1,1]$.
$\mathscr{K}$ is closed: Let a convergent sequence $f_n(x)$ from $\mathscr{K}$ be given, and let $f(x)$ be the limit. As the absolute value is a continuous function, we can let n tend to ∞ in the inequality $|f_n(x) - f_n(y)| < K\,|x-y|$ and get $|f(x) - f(y)| < K\,|x-y|$. Since the sequence $\{f_n x\}$ converges pointwise, $f(x) = 0$. Hence $\mathscr{K}$ is closed.
$\mathscr{K}$ is equicontinuous: Let $\epsilon >$ be a number and put $\delta = \epsilon/K$. Then by the inequality in the problem, we get

$$|f(x) - f(y)| \leq K\,|x-y| < K\delta = K\epsilon/K = \epsilon$$

which is vald for all x, y satisfying $|x-y| < \delta$ and all $f \in \mathscr{K}$.

PROBLEM 10: Any closed ball $\bar{B}(a;r)$ in $\mathbb{R}^n$ is closed and bounded, hence compact by the -Weierstrass theorem.

Let a closed ball B round 0 be given in $C([0,1],\mathbb{R})$, of radius r say. Let $f(x)$ be defined on $[0,1]$ by the usual "tent" construction:

$$f(x) = \begin{cases} 0 & x > \frac{2}{n} \\ -rn(x - \frac{2}{n}) & \frac{1}{n} < x \leq \frac{2}{n} \\ rnx & 0 \leq x \leq \frac{1}{n} \end{cases}$$

Then $d(f_n, 0) = \sup\{|f_n(x)| : x \in [0,1]\} = r$ so each f_n belongs to B. Clearly f_n converges pointwise to zero, but no subsequence can converge uniformly since $d(f_n, 0) = r$ for all n. hence B is not compact. □

PROBLEM 11:

a) The derivative of $\sin nx$ is $n \cos nx$. Hence the mean value theorem gives us a c between x and y such that

$$\sin nx - \sin ny = n \cos nc(x - y).$$

b) The family $\mathscr{S}$ is not equicontinuous, since if x and y belong to $[-\pi/4, \pi/4]$ (an interval where $\cos x$ exceeds $\sqrt{2}/2$ and hence $1/2$) and $x - y = \delta$, then

$$\sin nx - \sin ny \geq \delta n/2$$

and if ϵ is given, choosing $n > 2\epsilon/\delta$ for any $\delta > 0$ gives us

$$\sin nx - \sin ny > \epsilon.$$

c) A subfamily of an equicontinuous family obviously being equicontinuous, $C(I, J)$ can not be equicontinuous after what we just did.

d) No, $C([0, 1], [0, 1])$ for example, has many sequences without convergent subsequences, *e.g.,* the one we constructed in problem 10. □

PROBLEM 12: The mean value theorem gives us two numbers c_1 and c_2 between t and u such that $|f_i(t) - f_i(u)| = |f_i'(c_i)| \, |t - u|$ for $i = 1, 2$. Hence $d_{MH}(f(t), f(u)) \leq M \, |t - u|$. This shows that the the set of curves with bounded speed connecting A and B is an equicontinuous family: Given $\epsilon > 0$, use $\delta = \epsilon/M$.

Again by the mean value theorem, we have $|f_i(u) - f_i(a)| < |f_i'(c_i)| \, (b - a)$,for c_i's between a and u, and hence the curves stay within the (Manhattan) circel round A with radius $2M(b - a)$, and they form a bounded family. □

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