Ark5: Solutions

I have written down some very sketchy solutions to some of the exercises. I have only treated the ones given for the friday sessions, and it has been done rather hastily, so forgive me if there are errors. Still, I hope, they will be useful for you.

PROBLEM 1: We have that

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1 \end{cases}$$

Hence $f(1) = 1$. Now $f_n(x_n) = (1 - \frac{1}{n})^n$ which tends to e^{-1}. The sequence $\{f_n\}$ do not converge *uniformly* to f, hence lemma 3.5.7 does not apply. □

PROBLEM 2:

a) Let $\epsilon > 0$ be given. We have to produce an $\delta > 0$ such that $|f(n) - g(m)| < \epsilon$ once $d(n,m) < \delta$, and this for all $f \in C(X,\mathbb{R})$. But the metric on x is discete which means that $d(n,m) = 1$ if $n \neq m$; hence if we choose any δ less than one, $d(n,m) < \delta$ implies that $n = m$, and $|f(n) - f(m)| = 0 < \epsilon$.

b) As $\mathbb{R}^m$ is equipped with the sup-norm metric, $d\mathbb{R}^m(f,g) = \sup\{|f(i) - g(i)| : 1 \geq i \leq m\}$ which clearly is equal to $d(f,g) = \sup\{|f(x) - g(x)| : x \in X\}$ since $X = \{1, 2, 3, \ldots, m\}$.

c) Recall what the Bolzano-Weierstrass theorem says: Every bounded sequence in $\mathbb{R}^m$ has a convergent subsequence.

By 2.b) the map F gives a one to one corespondence between sequences in $\mathbb{R}^m$ and $C(X,\mathbb{R})$, and because F is an isometry, a sequence in $\mathbb{R}^m$ converges if and only if the corresponding sequence in $C(X,\mathbb{R})$ converges. Furthermore, a sequence in $\mathbb{R}^m$ is bounded if and only if the same holds for the corresponding sequence in $C(X,\mathbb{R})$.

Now, as the whole space $C(X,\mathbb{R})$ is equicontinuous, any sequence will also be. And hence every bounded sequence has a convergent subsequence by the A&A - theorem. And this exactly what B&W say. □

PROBLEM 3: First of all, if the function σ is constant its value must be zero since $\sigma(0) = 0$, and the functions having $\sigma = 0$ as modulus of continuity are all constant. Clearly a family of constant functions is equicontinuous (whatever ϵ is, use any δ you want).

If σ is not constant, the set $\{\sigma(x) : x \in [0,\infty)\}$ is an interval containing 0, but not reduced to $\{0\}$. This since σ is continuous an nonconstant. Hence for any $\epsilon >$ there is a $\delta > 0$ such that $\sigma(\delta) < \epsilon$.

a) Let $\mathscr{F}$ be a family of functions from X to Y all having σ as modulus of continuity. Let $\epsilon > 0$ be given and choose δ such that $\sigma(\delta) < \epsilon$. Then if $d_X(x, y) < \delta$ we get

$$d_X(f(x), f(y)) < \sigma(d_X(x, y)) \leq \sigma(\delta) < \epsilon$$

since σ is nondecreasing.

b) We want to apply the A&A theorem. We know that the family $\mathscr{K}$ is equicontinuous, and have only to check that it is closed and bounded.

Boundedness: We have the inequality:

$$|f(x)| = |f(x) - f(x_0)| \leq \sigma(d_X(x, x_0)). \qquad (*)$$

By assumption σ is continuous, and we know that $d_X(x, x_0)$ is a continuous function of x. Hence the composition $\sigma(d_X(x, x_0))$ is a continuos function of x. Since X is compact, it has maximal value, say M. Then $*$ translate into:

$$|f(x)| = |f(x) - f(x_0)| \leq \sigma(d_X(x, x_0)) \leq M.$$

and we conclude that $\mathscr{K}$ is bounded.

Closedness: Let f_n be a sequence in $\mathscr{K}$ converging to f. Clearly $f(x_0) = \lim_{n\to\infty} f_n(x_0) = 0$. Let $\epsilon > 0$ be given and choose N such that $|f_n(z) - f(z)| < \epsilon$ whenever $n \geq N$ and for all $z \in X$; which can be done since f_n tends uniformly to f. The we get:

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < 2\epsilon + \sigma(d(x, y)).$$

This holds for all $\epsilon > 0$, and it follows that

$$|f(x) - f(y)| \leq \sigma(d(x, y)).$$

Concequently $f \in \mathscr{K}$, that is, $\mathscr{K}$ is closed. □

PROBLEM 4:

a) A polynomial is shaped like $p(x) = \sum_{k=0}^{n} a_k x^k$ where the a_k's are constants. This gives

$$\int p(x)f(x)\,dx = \sum_{k=1}^{n} a_k \int x^k f(x)\,dx = 0$$

since for each k, $\int x^k f(x)\,dx = 0$.

b) Vi have for any polynomial $p(x)$:

$$\int f(x)(f(x) - p(x))\,dx = \int f(x)^2\,dx - \int f(x)p(x)\,dx = \int f(x)^2\,dx \qquad (*)$$

By Weierstrass' approximation theorem we can, for each given $\epsilon > 0$, find a polynomial such that $|f(x) - p(x)| < \epsilon$ for all $x \in [a,b]$. Integration gives

$$\int |f(x)(f(x) - p(x))| \, dx \leq M(b-a)\epsilon,$$

where $M = \sup\{|f(x)| : x \in [a,b]\}$. Hence by ❋ $\int f(x)^2 \, dx < M(b-a)\epsilon$, but then $\int f(x)^2 \, dx = 0$ since $\epsilon > 0$ can be choosen freely. This implies that $f(x) = 0$ since $f(x)^2$ is a positive function. □

PROBLEM 5:
a) If $f(x)$ is continuously diffentiable, we get by partial integration:

$$\int_{-\pi}^{\pi} f(x) \cos nx \, dx = |_{-\pi}^{\pi} f(x) \frac{\sin nx}{n} - \frac{1}{n} \int_{-\pi}^{\pi} f'(x) \sin nx \, dx = -\frac{1}{n} \int_{-\pi}^{\pi} f'(x) \sin nx \, dx$$

hence

$$\left| \int_{-\pi}^{\pi} f(x) \cos nx \, dx \right| \leq \left| \frac{1}{n} \int_{-\pi}^{\pi} f'(x) \sin nx \, dx \right| \leq \frac{2\pi M}{n} \tag{❂}$$

where $M = \sup\{|f'(x)| : x \in [-\pi, \pi]\}$ which is finite since f' is continuous. Now as $n \to \infty$, the right side of ❂ tends to zero.

b) By Weierstrass approximation theorem we may find, for each $\epsilon > 0$ given, a polynomial $p(x)$ shuch that $|f(x) - p(x)| < \epsilon/2$ for all $x \in [-\pi, \pi]$. Since $p(x)$ is continuously diffentiable, we may by 5.a) finde an N such that for $n > N$ then $\left|\int p(x) \cos nx \, dx\right| < \epsilon/2$. This gives

$$\left| \int_{-\pi}^{\pi} f(x) \cos \right| nx \, dx \leq \int_{-\pi}^{\pi} |f(x) - p(x)| \, dx + \left| \int p(x) \cos nx \, dx \right| < \epsilon/2 + \epsilon/2 = \epsilon$$

whenever $n > N$. □

PROBLEM 6: Vi have to assume that $f(x)$ is a *continuous* function on $[-\pi, \pi]$ satisfying $f(-x) = f(x)$.

The function $\arccos t$ is well defined on $[-1,1]$ and satisfies $\arccos(\cos x) = x$ for $x \in [0, \pi]$. Hence $f(\arccos t)$ is a continuous function on $[-1,1]$. For each n there is — after Weierstrass — a poly $p_n(t)$ defined in $[-1,1]$ such that $|f(\arccos t) - p_n(t)| < \frac{1}{n}$ for all $t \in [-1,1]$. Putting $t = \cos x$ we get

$$|f(x) - p_n(\cos x)| < \frac{1}{n} \tag{✱}$$

for all $x \in [0, \pi]$ (since $\arccos(\cos x) = x$ for $x \in [0, \pi]$). Now both $f(x)$ and $\cos x$ are even functions, so the inequallity ✷ also holds for x in $[-\pi, 0]$. The sequence $p_n(\cos x)$ the tends to $f(x)$ as $n \to \infty$. □

PROBLEM 7: Assume that there is a sequence of polynomials $\{P_n(x)\}$ that converges uniformly to $\frac{1}{x}$ in $(0, 1)$. Let $\epsilon > 0$ be a number. There is an N such that if $n \geq N$, then $\left|\frac{1}{x} - P_n(x)\right| < \epsilon$ for all $x \in (0, 1)$. This implies that $\frac{1}{x} < P_n(x) +$ for all $x \in (0, 1)$ which is absurd since the right side of the inequality is bounded whereas the left side is unbounded. □

PROBLEM 8: Assume that there is a sequence of polynomials $\{P_n(x)\}$ that converges uniformly to e^x in $\mathbb{R}^+$. Pick any $\epsilon > 0$ and find a N such that if $n > N$, then $|e^x - P_n(x)| < \epsilon$ for all $x \in \mathbb{R}^+$. This implies that $e^x < P_n(x) + \epsilon$ and hence $1 < e^{-x}P_n(x) + e^{-x}\epsilon$ for all $x \in \mathbb{R}^+$, which is absurd since the right side of the inequality tends to zero.

If we replace e^x by e^{-x} we get an inequality $|P_n(x) - e^{-x}| < \epsilon$ for all $x \in \mathbb{R}^+$, but e^{-x} tends to zero when $x \to \infty$ whereas $P_n(x)$ tends to ∞ or $-\infty$, so they cannot be only an ϵ apart. □

PROBLEM 9:
a) We use induction on n and assume that $f^{(n)}(x) = e^{-1/x^2}\frac{P_n(x)}{x^{N_n}}$ where $P_n(x)$ is some polynomial. We get when we differentiat:

$$\begin{aligned} f^{(n+1)}(x) =& e^{-1/x^2}\frac{x^{N_n}P_n'(x) - N_n x^{N_n-1}P_n(x)}{x^{2N_n}} - e^{-1/x^2}\frac{2x}{x^4}\cdot\frac{p_n(x)}{x_{N_n}} \\ & e^{-1/x^2}\frac{x_{N_n}P_n'(x) - (N_n x^{N_n-1} + 2x^{N_n})P_n(x)}{x^{N_n+3}} \end{aligned}$$

which certainly is of the form $e^{-1/x^2}\frac{P_{n+1}(x)}{x^{N_{n+1}}}$ with $P_{n+1}(x) = x_{N_n}P_n'(x) - (N_n x^{N_n-1} + 2x^{N_n})P_n(x)$ and $N_{n+1} = N_n + 3$.
b) As $\lim_{y\to\infty} e^{-y^2}Q(y) = 0$ for any rational function $Q(y)$, we see that $\lim_{x\to 0} f^{(n)}(x) = 0$ for all n. This means that f has infinitely many derivatives in zero, and that they all take the value 0 there.
c) The Taylor series at the origin of f being $\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$, it follows by what we just did, that the Taylor series is identically zero. But the function e^{-1/x^2} is not! □

PROBLEM 11: The binomial formula gives us

$$(xe^{\alpha/n} + x - 1)^n = \sum_{r=0}^{n} (e^{\alpha/n})^r x^r (x-1)^{n-r} = \sum_{r=0}^{n} e^{r\alpha/n} x^r (x-1)^{n-r}$$

and if $f(x) = e^{\alpha x}$ then $f(\frac{r}{n}) = e^{r\alpha/n}$. □

PROBLEM 11: The binomial formula gives us

$$(xe^{\alpha/n} + x - 1)^n = \sum_{r=0}^{n} (e^{\alpha/n})^r x^r (x-1)^{n-r} = \sum_{r=0}^{n} e^{r\alpha/n} x^r (x-1)^{n-r}$$

and if $f(x) = e^{\alpha x}$ then $f(\frac{r}{n}) = e^{r\alpha/n}$. □

Versjon: Monday, March 26, 2012 3:11:13 PM