Ark1: Exercises for MAT2400 — Metric spaces

Mathematicians like to make general theories. It saves work to do things once and for all, but more importantly, often it makes things clearer! It turns the spotlight on the salient points of the theory, and shows what is the important questions. Irrelevant and disturbing elements are removed.

In the course and in some of the exercises, we will from time to time work with rather peculiar metrics. This is to build an intuition of what is true and what is not in metric spaces, and also, of course, to make you familiar with the new concepts that are introduced.

So, even if our main reason to study metric spaces is their use in the theory of function spaces (spaces which behave quite differently from our old friends $\mathbb{R}^n$), it is useful to study some of the more exotic spaces.

The exercises on this sheet covers the pharagraphs **12.1** to **12.3** in the book.

Some of the exercises are from Tom Lindstrøm's excellent notes from last year, and for your convenience they are copied on this sheet (a praxis I do not guarantee will continue). They are referred to as "*Tom's notes x.x, Problem z (page y)*". A new and even better version of Tom's notes is now on our web side, and we will rely on that during the whole semester.

Key words: Metric spaces, convergence of sequences, equivalent metrics, balls, open and closed sets, exterior points, interior points, boundary points, induced metric.

Examples of metrics, elementary properties and new metrics from old ones

PROBLEM 1. Show that the Manhatten metric (or the taxi-cab metric; example **12.1.7** in the book), the sup-norm metric (example **12.1.9**) and the Hamming metric all are metrics.

PROBLEM 2. *(Tom's notes 2.1, Problem 24 (page 7)).* Show that if $d(x, y)$ is a metric on X, then

$$|d(x, y) - d(y, z)| \leq d(x, z)$$

for all x, y and z in X.

PROBLEM 3. Show that $d(f, g) = \sup\{|f(x) - g(x)| \mid x \in [a, b]\}$ is a metric on the space $C([a, b])$ of continuous functions on the closed interval $[a, b]$.

PROBLEM 4. *(Tom's notes 2.1, Problem 23 (page 5)).* A sequencs $\{x_n\}$ of real numbers is called *bounded* if there is a number M such that $|x_n| \leq M$ for all n. Let X be the set of all bounded sequences, and let $d(\{x_n\},\{y_n\}) = \sup\{|x_n - y_n| \mid n \in \mathbb{N}\}$. Show that d is a metric on X.

Problem 5. Let X be the set of all absolutely convergent series of real numbers; *i.e.,* series $\{a_n\}$ with $a_n \in \mathbb{R}$ such that $\sum_{n\geq 0} |a_n| < \infty$. If $\alpha = \{a_n\}$ and $\beta = \{b_n\}$ are two elements in X (*i.e.,* two absolutely convergent series), we let

$$d(\alpha, \beta) = \sum_{n\geq 0} |a_n - b_n|.$$

Show that d is a metric on X.

Problem 6. The triangle inequality is valid in a metric space X with metric $d(x, y)$: $d(x, y) \leq d(x, z) + d(z, y)$. It is a natural question — and it is from time to time usefull to know — when equality holds. What relations must there be between x, y and z for $d(x, y) = d(x, z) + d(z, y)$ to be true? Answere that question in each of the following cases:

a) $X = \mathbb{R}$ with standard metric $|x - y|$.

b) $X = \mathbb{R}^2$ with standard metric $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.

c) $X = \mathbb{R}^2$ with the Manhatten metric $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$.

d) $X = \mathbb{R}^2$ with the sup-norm metric $d(x, y) = \sup\{|x_1 - y_1|, |x_2 - y_2|\}$.

Problem 7. In this exercise f is a strongly increasing function on $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\}$ satisfying $f(0) = 0$.

a) Show that if f is subadditive, *i.e.,* satisfies $f(x+y) \leq f(x)+f(y)$ for all $x, y \in \mathbb{R}^+$, then $f(d(x, y))$ is a metric on X.

b) Show that if f is continuous on $\mathbb{R}^+$ and twice differentiable with $f''(x) \leq 0$ for all $x > 0$, then f is subadditive. (Hint: show that $f(x+b) - f(x)$ is decreasing on $\mathbb{R}^+$ for all $b \in \mathbb{R}^+$).

c) Let $d(x, y)$ be a metric on the set X. Show that also $\frac{d(x,y)}{1+d(x,y)}$ is a metric on X.

Problem 8.

a) For which functions $f(x)$ is $d(x, y) = |f(x) - f(y)|$ a metric on $\mathbb{R}$?

b) Show that $|\frac{1}{x} - \frac{1}{y}|$ defines a metric on $\mathbb{R} \setminus \{0\}$.

Problem 9. *(Tom's notes 2.1, Problem 24 (page 6)).* Let V be a (real) vector space, a function $|\cdot|\colon V \to \mathbb{R}$ is called a *norm* if the following conditions are satisfied:

(i) For all $v \in V$, $|v| \geq 0$, with equality if and only if $v = 0$.

(ii) $|\alpha v| = |\alpha||v|$ for all $\alpha \in \mathbb{R}$ and all $v \in V$.

(iii) $|v + w| \leq |v| + |w|$ for all $v, w \in V$.

Show that if $|\cdot|$ is a norm, then $d(v, w) = |v - w|$ defines a metric on V.

Equivalent metrics

We call two metrics d_1 and d_2 on a set X *equivalent* if there are constants M_1 and M_2 satisfyng $d_1(x, y) \leq M_1 d_2(x, y)$ and $d_2(x, y) \leq M_2 d_1(x, y)$ for all points $x, y \in X$ (see *Tom's notes 2.2, Problem 27 (page 7)*).

PROBLEM 10. If d_1 are d_2 two equivalent metrics on X show that a sequence $\{a_n\}$ converges when we equip X with d_1 if and only if it converges when we equip X with d_1. Are the two limits equal?

PROBLEM 11. If X is a finite set, show that all metrics on X are equivalent.

PROBLEM 12. Show that the Manhattan metric, the sup-norm metric and the standard, Euclidian metric on $\mathbb{R}^n$ are equivalent.

Balls, open and closed sets

PROBLEM 13. For each of the three cases, the Manhattan metric, the sup-norm metric and the standard metric on $\mathbb{R}^2$, describe and draw the ball $B((0,0); 1)$.

PROBLEM 14. Show that two equivalent metrics on a set X defines exactly the same open and closed sets. Do they define the same open balls?

PROBLEM 15. Let X be a metric space with metric $d(x, y)$. Show that a set $\{x\}$ with just one element is closed. What about sets with exactly two elements?

PROBLEM 16. Let $t \in \mathbb{R}$ be a real number, and let $A = \{(x, y) \in \mathbb{R}^2 \mid y^2 \leq x \text{ and } x > t\}$.

a) For $t = 1$, $t = 0$ and $t = -1$ decide in each case if A is open, closed or neither.

b) In each of the three cases above, describe the interior points of A, the exterior points of A and its boundary points.

c) In each of the three cases, what is the closure of A?

PROBLEM 17. Let $X = \{x \in \mathbb{R} \mid |x| > 5\}$ with the metric $d(x, y)$ induced from the standard metric on $\mathbb{R}$. Find *all* subsets of X that are both open and closed (as subsets of the metric space (X, d)).

PROBLEM 18. Let X be the space of continuous functions on $[0, 1]$ with the sup-norm metric (*i.e.*, $d(f, g) = \sup\{|f(x) - g(x)| \mid x \in [0, 1]\}$). Show that the subset

$$A = \{f \in X \mid f(x) > 1, \text{ for } x \in [1/3, 2/3]\}$$

is an open set. Is the subset

$$B = \{f \in X \mid f(x) > 1, \text{ for } x \in (1/3, 2/3)\}$$

open, closed or neither?

PROBLEM 19. Let $C([a, b])$ be the space of continuous functions on the closed interval $[a, b]$ equipped with the metric $d(f, g) = \int_a^b |f(x) - g(x)|\, dx$. Show that the set of functions $f \in C([a, b])$ satisfying $\int_a^b f(x)\, dx > 1$ is an open set.

PROBLEM 20. *(Tom's notes 2.3, Problem 34 (page 11)).* Let (X, d) be a metric space. Show that the union of any collection of open sets is open. Show that the intersection of a *finite* collection of open sets is open. Give an example of a collection $G_1, G_2, \ldots$ of open sets whose intersection is not open.

PROBLEM 21. *(Tom's notes 2.3, Problem 34 (page 12)).* Let (X, d) be a metric space. Show that the intersection of any collection of closed sets is closed. Show that the union of a *finite* collection of closed sets is closed. Give an example of an infinit collection $F_1, F_2, \ldots$ or closed sets whose union is not closed.

PROBLEM 22. (This is basically *Exercise 12.2.4 in the book.*) Let $X = \{(0, y) \in \mathbb{R}^2 \mid y > 1/2\} \cup \{(x, 0) \in \mathbb{R}^2 \mid |x| < 10\}$ and fit X out with the metric $d(x, y)$ induced from the standard metric on $\mathbb{R}^2$.

a) Find a ball $B(a; r)$ in X such that the closure of the open ball $B(a; r)$ is *not* equal to the closed ball $\bar{B}(a; r) = \{b \in X \mid d(b, a) \leq r\}$. How many balls like that can you find in X? HINT: Make a drawing of X.

b) Show that in general, for any metric space (X, d), the closure of $B(a; r)$ is contained in the closed ball $\bar{B}(a; r)$.

PROBLEM 23. Let $d(x, y)$ be a metric on X and define $e(x, y) = \frac{d(x,y)}{1+d(x,y)}$. Denote by $B_e(a; r)$ and $B_d(a; r)$ the open balls round a with radius r in the two metric spaces (X, d) and (X, e).

a) Show that $B_e(a; r) = B_d(a; \frac{r}{1-r})$ for $0 < r < 1$.

b) Show by an example that the two metrics e and d are not necessarily equivalent.

Subset metrics

PROBLEM 24. *(Tom's notes 2.3, Problem 33 (page 8 and 9)).* Let (X, d) denote a metric space, and let $A \subseteq X$ be a subset. We shall use the subset metric d_A on A.

a) If $G \subseteq A$ is open (resp. closed) in X, then it is open (resp. closed) in A.

b) Show by exhibiting an example, that $G \subseteq A$ might be open (resp. closed) in A without being open (resp. closed) in X.

c) Assume that A is open (resp. closed) in X. Then a subset $G \subseteq A$ is open (resp. closed) in A if and only if it is open (resp. closed) in X.