## Ark13: Exercises for MAT2400 — Integrals of nonnegative functions

The exercises on this sheet cover the section *5.6.* They are intended for the groups on Thursday, May 10 and Friday, May 11.

The distribution is the following: *Friday, May 11:* No 1, 2, 3, 6, 7, 8, 13.
The rest for Thursday, May 10.

**Key words**: Integrals of nonnegative functions, and repetition of metric spaces.

### Integrals of nonnegative functions

PROBLEM 1. *(Chebychev's inequality; Tom's notes 5.6, Problem 3 (page 175)).* Let $f\colon \mathbb{R}^d \to [0,\infty]$ be a measurable function. Show that for any positive, real number $a$, it holds true that

$$\mu(\{x : f(x) \geq a\}) \leq a^{-1} \int f \, d\mu.$$

PROBLEM 2. Let $f$ be a nonnegative, integrable function on $\mathbb{R}^d$; *i.e.,* a nonnegative measurable function such that $\int f \, d\mu < \infty$.

a) Given $\epsilon > 0$, show that there is a real number $M$ such that

$$\mu(\{x : f(x) \geq M\}) < \epsilon.$$

b) Show that

$$\mu(\{x : f(x) = \infty\}) = 0.$$

PROBLEM 3. Assume that $f\colon \mathbb{R}^d \to [0,\infty]$ is an integrable function. Show that for any given $\epsilon > 0$, we may write $f = f_1 + f_2$ where $f_1$ and $f_2$ are two measurable functions with values in $[0,\infty]$ such that

i) $f_1$ is bounded,

ii) $\int f_2 \, d\mu < \epsilon$.

HINT: Approximate $f$ by simple functions.

PROBLEM 4. *(Tom's notes 5.6, Problem 4 (page 175)).* Assume that $f$ and $g$ are two measurable functions on $\mathbb{R}^d$ taking values in $[0,\infty]$. Show the following three statements:

a) The function $f$ satisfies $f = 0$ a.e. if and only if $\int f\, d\mu = 0$.
b) If $f = g$ a.e., then $\int f\, d\mu = \int g\, d\mu$.
c) If $\int_E f\, d\mu = \int_E g\, d\mu$ for all measurable subsets $E \subseteq \mathbb{R}^d$, then $f = g$ a.e.

Problem 5. *(Tom's notes 5.6, Problem 5 (page 175)).* We assume that $f\colon \mathbb{R}^d \to [0, \infty]$ is a measurable function and that $A$ and $B$ are two subsets of $\mathbb{R}^d$. Show the following statements:
a) If $A \subseteq B$, then $\int_A f\, d\mu \leq \int_B f\, d\mu$. What can you say if $A \subset B$?
b) If $A \cap B = \emptyset$, then

$$\int_{A\cup B} f\, d\mu = \int_A f\, d\mu + \int_B f\, d\mu.$$

c) In general,

$$\int_{A\cup B} f\, d\mu + \int_{A\cap B} f\, d\mu = \int_A f\, d\mu + \int_B f\, d\mu.$$

**Hint:** Take a look at problem 8 on Ark12.

Problem 6. *(Tom's notes 5.6, Problem 7 (page 175)).* Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{otherwise ,} \end{cases}$$

in other words, $f(x)$ is the characteristic function $\chi_{\mathbb{Q}}$ of the rationals. For each $n \in \mathbb{N}$, let $f_n\colon \mathbb{R} \to \mathbb{R}$ be given by

$$f_n(x) = \begin{cases} 1 & \text{if } x = \frac{p}{q} \text{ where } p \in \mathbb{Z} \text{ and } q \in \mathbb{N} \text{ are relatively prime and } q \leq n \text{ ,} \\ 0 & \text{otherwise .} \end{cases}$$

a) Show that $\{f_n(x)\}$ is an increasing sequence converging to $f(x)$ for all $x \in \mathbb{R}$.
b) Show that each function $f_n(x)$ is Riemann integrable over $[0, 1]$ and that $\int_0^1 f(x)\, dx = 0$.
c) Show that $f(x)$ is *not* Riemann integrable over $[0, 1]$.
d) Show that the Lebesque integral $\int_{[0,1]} f\, d\mu$ exists and find its value.

Problem 7. *(Tom's notes 5.6, Problem 8 (page 176)).*
a) Let $\{u_n(x)\}$ be a sequence of positive, measurable functions. Show that

$$\int \sum_{n=1}^{\infty} u_n\, d\mu = \sum_{n=1}^{\infty} \int u_n\, d\mu.$$

b) Assume that $f$ is a nonnegative measurable function and that $\{B_n\}$ is a sequence of disjoint, measurable sets with $B = \bigcup_{n=1}^{\infty} B_n$. Show that

$$\int_B f\,d\mu = \sum_{n=1}^{\infty} \int_{B_n} f\,d\mu.$$

PROBLEM 8. *(Tom's notes 5.6, Problem 9 (page 176)).* Let $f$ be a nonnegative measurable function and $\{A_n\}$ an increasing sequence of measurable sets whose union we denote by $A$. Show that

$$\int_A f\,d\mu = \lim_{n\to\infty} \int_{A_n} f\,d\mu.$$

PROBLEM 9. Let $f$ be a nonnegative measurable function and let $\{A_n\}$ be a decreasing sequence of measurable sets whose union we denote by $A$. Assume that $\int_{A_1} f\,d\mu < \infty$. Show that

$$\int_A f\,d\mu = \lim_{n\to n} \int_{A_n} f\,d\mu$$

PROBLEM 10. *(Tom's notes 5.6, Problem 11 (page 176)).* Find a decreasing sequence $\{f_n(x)\}$ of measurable functions $f_n\colon \mathbb{R} \to [0,\infty)$ converging pointwise to zero, but such that $\lim_{n\to\infty} \int f_n(x)\,d\mu \neq 0$.

PROBLEM 11. *(Tom's notes 5.6, Problem 13 (page 176)).* Assume that $\{f_n(x)\}$ is a sequence of nonnegative measurable functions on $\mathbb{R}^d$ converging pointwise to $f(x)$. Show that if

$$\lim_{n\to\infty} \int f_n\,d\mu = \int f\,d\mu < \infty,$$

then

$$\lim_{n\to\infty} \int_E f_n\,d\mu = \int_E f\,d\mu < \infty$$

for all measurable subsets $E \subseteq \mathbb{R}^d$.

PROBLEM 12. *(Tom's notes 5.6, Problem 14 (page 177)).* Assume that $g\colon \mathbb{R}^d \to [0,\infty]$ is an integrable function and that $\{f_n\}$ is a sequence of nonnegative, measurable functions converging pointwise to $f$. Show that if $f_n \le g$ for all $n$, then

$$\lim_{n\to\infty} \int f_n\,d\mu = \int f\,d\mu.$$

HINT: Apply Fatou's lemma to both sequences $\{f_n\}$ and $\{g - f_n\}$.

## Repetition — metric spaces.

Problem 13. Problem 2 of the mandatory assignment I from last year. You find it on the homepage of MAT2400, spring 2011.

Problem 14. Problem 3 of the mandatory assignment I from last year. You find it on the homepage of MAT2400, spring 2011.

Versjon: Thursday, May 3, 2012 9:01:09 AM