Ark14: Exercises for MAT2400 — Integrals of nonnegative functions

The exercises on this sheet cover the section *5.7* and *5.8*. They are intended for the groups on Thursday, May 24 and Friday, May 25.

The distribution is the following: *Friday, May 25:* No 3 ,8 , 9, 10, 12.
The rest for Thursday, May 24.

Key words: Integrable functions. Repetition.

Integrable functions and Fourier coefficients

PROBLEM 1. *(Riemann-Lebesque lemma for simple functions).* The aim of this exercise is to show that if $f(x)$ is a simple function on $[-\pi, \pi]$, then $\lim_{n\to\infty} \int_{-\pi}^{\pi} f(x) \cos nx \, d\mu = 0$. (It is of course true that corresponding *sinus*-integral also satisfies $\lim_{n\to\infty} \int_{-\pi}^{\pi} f(x) \sin nx \, d\mu = 0$, but we will consentrate on the *cosinus*-variant.)

a) If $I = [a, b] \subseteq [-\pi, \pi]$, then $\lim_{n\to\infty} \int_{-\pi}^{\pi} \chi_I \cos nx \, d\mu = 0$.

b) Show that if $E \subseteq (-\pi, \pi)$ is measurable, then for any $\epsilon > 0$ there is a set G which is a finite union of disjoint, open intervals contained in $[-\pi, \pi]$ satisfying $E \subseteq G$ and $\mu(G \setminus E) < \epsilon$. HINT: Take a look at **Proposition 5.3.5** in Tom's.

c) Show that if $E \subseteq (-\pi, \pi)$, then $\lim_{n\to\infty} \int_{-\pi}^{\pi} \chi_E \cos nx \, d\mu = 0$, and use that to show that for any *simple* function $\phi(x)$ defined in $[-\pi, \pi]$ it holds true that

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \phi(x) \cos nx \, d\mu = 0.$$

PROBLEM 2. *(Riemann-Lebesque lemma for integrable functions).* Show that if $f(x)$ is an *integrable* function in $[-\pi, \pi]$, then

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} f(x) \cos nx \, d\mu = 0.$$

HINT: Approximate f by simple functions.

PROBLEM 3. Throughout this problem, we fix a sequence a_k of real numbers and an increasing sequence n_k of natural numbers with $\lim_{k\to\infty} n_k = \infty$.

Let $\beta > 0$ be given, and define $E_\beta = \{x \in [-\pi, \pi] : \lim_{n\to\infty} \cos(n_k x + a_k) = \beta\}$. Show that $\mu(E_\beta) = 0$. HINT: Use Lebegues Dominated Convergence Theorem and the Riemann-Lebesque lemma. If $\beta = 0$, the identity $\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x$ may be useful.

Integrable functions

PROBLEM 4. *(Reverse Fatou).* Let $\{f_n\}$ be a sequence of extended real valued measurable functions on $\mathbb{R}^d$ (*i.e.*, measurable functions $f_n\colon \mathbb{R}^d \to \bar{\mathbb{R}}$). Assume that there is an integrable function g on $\mathbb{R}^d$ such that $f_n \le g$ for all n. Show that then

$$\limsup_{n\to\infty} \int f_n \, d\mu \le \int \limsup_{n\to\infty} f_n \, d\mu.$$

PROBLEM 5. *(Tom's notes 5.7, Problem 5 (page 181)).* Assume that $f\colon \mathbb{R}^d \to \bar{\mathbb{R}}$ is a measurable function.

a) Show that if f is integrable over the measurable set A, and if A is the uion of an increasing sequence of measurable sets A_n, then

$$\lim_{n\to\infty} \int_{A_n} f \, d\mu = \int_A f \, d\mu.$$

b) Assume that $\{B_n\}$ is a decreasing sequence of measurable sets with intersection B. Show that if f is integrable over B_1, then

$$\lim_{n\to\infty} \int_{B_n} f \, d\mu = \int_B f \, d\mu.$$

PROBLEM 6. *(Basically Tom's notes 5.7, Problem 6 (page 181)).* Show that if $f\colon \mathbb{R}^d \to \bar{\mathbb{R}}$ is integrable over a measurable set A, and $\{A_n\}$ is a sequence of disjoint, measurable sets whose union is A, then

$$\int_A f \, d\mu = \sum_{n=1}^{\infty} \int_{A_n} f \, d\mu.$$

PROBLEM 7. *(Basically Tom's notes 5.7, Problem 7 (page 181)).* Assume that $f\colon \mathbb{R} \to \bar{\mathbb{R}}$ is an integrable function. Define for each natural number n the set $A_n = \{x \in \mathbb{R} : |f(x)| \ge n\}$.

a) Show that

$$\lim_{n\to\infty} \int_{A_n} f \, d\mu = 0.$$

b) Show that for any $\epsilon > 0$, one may decompose f as a sum $f = g + h$ of two integrable functions, with g bounded, and with h satisfying $\int |h| \, d\mu < \epsilon$.

PROBLEM 8. Assume that $f\colon \mathbb{R} \to \bar{\mathbb{R}}$ is an integrable function, and let $a \in \mathbb{R}$. Let $F(x) = \int_{[a,x]} f\, d\mu$ (which one also would like to denote by $\int_a^x f\, d\mu$). Show that $F(x)$ is continuous.

PROBLEM 9. *(Tom's notes 5.7, Problem 8 (page 182), Extended version of the Dominated Convergence theorem).* Let $g\colon \mathbb{R}^d \to \bar{\mathbb{R}}$ be a nonnegative, integrable function, and let $\{f_n\}$ be a sequence of measurable functions converging a.e. to f. Assume that $|f_n(x)| \leq g(x)$ for almost all x, then, show that

$$\lim_{n\to\infty} \int f_n\, d\mu = \int f\, d\mu.$$

Repetisjon

PROBLEM 10. Eksamen MAT1300, 1. juni 2007 No. 3.

PROBLEM 11. Eksamen MAT1300, 1. juni 2007 No. 4.

PROBLEM 12. Eksamen MAT1300, 1. juni 2006 No.1: a), b), c) and e).

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