Ark5: Exercises for MAT2400 — Weierstrass' approximation theorem

The exercises on this sheet cover the sections *3.7* Tom's notes, but with a few more exercises from *3.5*. They are for the groups on Thursday, Mars 1 and Friday, Mars 2. With the following distribution:

Thursday, Mars 1: No 2, 3, 7, 9, 10, 12.
The rest for Friday

Key words: Uniform approximation by polynomials, Weierstrass' approximation theorem, Bernstein polynomials.

Arzelà - Ascoli

PROBLEM 1. Consider the sequence $f_n(x) = x^n$ of functions on $[0,1]$ and the sequence of points $x_n = 1 - \frac{1}{n}$. Show that $\{f_n\}$ converges pointwise to a function f and determine the limit function f. Show that $f_n(x_n) \neq f(1)$. Relate this to **Lemma 3.5.7** in Tom's.
HINT: Revivement of old knowlegde: $\lim_{n\to\infty}(1-\frac{1}{n})^n = e^{-1}$.

PROBLEM 2. Let $m \in \mathbb{N}$ and let $X = \{1, 2, \ldots, m\}$ equipped with the discrete metric.

a) Show that the whole space $C(X,\mathbb{R})$ is equicontinuous.

b) Show that the map $F\colon C(X,\mathbb{R}) \to \mathbb{R}^m$ given by $F(f) = (f(1), f(2), \ldots, f(m))$ is an isometry[1] when we equip $\mathbb{R}^m$ with the sup-norm metric.

c) Show that the Arzelà-Ascoli theorem in this setting is just the Bolzano-Weierstrass theorem.

PROBLEM 3. *(Tom's notes 3.5, Problem 6 (page 67)).* Assume that (X, d_X) and (Y, d_Y) are two metric spaces, and let $\sigma\colon [0,\infty) \to [0,\infty)$ be a nondecreasing, continuous function such that $\sigma(0) = 0$. We say that σ is *a modulus of continuity* for a function $f\colon X \to Y$ if

$$d_Y(f(u), f(v)) \leq \sigma(d_X(u,v))$$

for all $u, v \in X$.

a) Show that a family of functions with the same modulus of continuity is equicontinuous.

[1] This means that F is bijective and preserves distances

b) Assume that (X, d_X) is compact, and let $x_0 \in X$. Show that if σ is a modulus of continuity, then the set

$$\mathscr{K} = \{f\colon X \to \mathbb{R}^n : f(x_0) = 0 \text{ and } \sigma \text{ is a modulus of continuity for } f\ \}$$

is compact.

c) Show that every function in $C([a, b], \mathbb{R}^m)$ has a modulus of continuity.

Weierstrass' approximation theorem

PROBLEM 4. *(Tom's notes 3.7, Problem 4 (page 76)).* Assume that f is a continuous real valued function on $[a, b]$ such that $\int_a^b x^k f(x)\,dx = 0$ for $k = 0, 1, 2, \ldots$.

a) Show that $\int_a^b p(x)f(x)\,dx = 0$ for all polynomials p.

b) Use Weierstrass' approximation theorem to show that $\int_a^b f(x)^2\,dx = 0$. Conclude that $f(x) = 0$ for all $x \in [a, b]$,

PROBLEM 5. Let $f(x)$ be a continuous function on the inerval $[0, 2\pi]$. Show that

$$\lim_{n\to 0} \int_0^{2\pi} f(x)\cos nx\,dx = 0. \qquad (\bigstar)$$

HINT: First, use partial integration to show that ★ holds whenever f is continuously differentiable. Then use the Weierstrass approximation theorem to treat the general case.

PROBLEM 6. Let $f(x)$ be a function defined on $[-\pi, \pi]$ satisfying $f(-x) = f(x)$ for all $x \in [-\pi, \pi]$. Show that there is a sequences of polynomials $P_n(t)$ such that $P_n(\cos x)$ converges uniformly to $f(x)$ on $[-\pi, \pi]$. HINT: Use Weierstrass' approximation theorem on the function $f(\arccos t)$ for $t \in [-1, 1]$.

PROBLEM 7. *(Tom's notes 3.7, Problem 1 (page 76)).* Show that there is no sequence of polynomials that converges uniformly to $\frac{1}{x}$ on the interval $(0, 1)$.

PROBLEM 8. *(Tom's notes 3.7, Problem 2 (page 67)).* Show that there is no sequence of polynomials that converges uniformly to e^x on $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$.

Is there a sequence of polynomials that converges uniformly to e^{-x} on $\mathbb{R}^+$?

PROBLEM 9. *(Tom's notes 3.7, Problem 3 (page 76)).* This exercise illustrates why Taylor polynomials are no substitute for Weierstrass' approximation.

We let the function f be defined as

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

a) Show that for $x \neq 0$ the n-th derivative of f has the form

$$f^n(x) = e^{-1/x^2} \frac{P_n(x)}{x^{N_n}},$$

where $P_n(x)$ is a polynomial and N_n is som natural number. HINT: Use induction on n.

b) Show that $f^n(0) = 0$ for all $n \in \mathbb{N}$.

c) Show that the Taylor polynomials of f at 0 do not converge to f except in the point 0.

PROBLEM 10. *(Tom's notes 3.7, Problem 5 (page 76)).* The aim of this exercise is to show that $C([a, b])$ with the sup-norm metric is a separable metric space.

a) Assume that (X, d) is a metric space and that $S \subseteq T$ are two subsets. Show that if S is dense in (T, d_T) (T with induced metric) and T is dense in X, then S is dense in X.

b) Show that for any polynomial p, there is a sequence of polynomials q_n with *rational* coefficients that converge uniformly to p on $[a, b]$.

c) Show that the subset of $C([a, b])$ consisting of polynomials with rational coefficients is dense.

d) Show that $C([a, b])$ is separable.

Bernstein polynomials

PROBLEM 11. Show that for any $\alpha \in \mathbb{R}$ and any natural number n the equality $B_n(e^{\alpha x}; x) = (xe^{\alpha/n} + 1 - x)^n$ is true.

PROBLEM 12. Let $f(x)$ be continuously differentiable on $[0, 1]$. Let $B_n(f; x)$ be the n-th Bernstein polynomial of f, *i.e.,* we have

$$B_n(f; x) = \sum_{r=0}^{n} f(\frac{r}{n}) \binom{n}{r} x^r (1 - x)^{n-r}.$$

a) Show that the following equality for the derivative $B'_n(f; x)$ holds true:

$$B'_{n+1}(f; x) = (n + 1) \sum_{r=0}^{n} \Delta f(\frac{r}{n+1}) \binom{n}{r} x^r (1 - x)^{n-r},$$

where $\Delta f(t) = f(t + \frac{1}{n+1}) - f(t)$.

b) Use the mean value theorem to show that there are points c_r beteween $\frac{r}{n+1}$ and $\frac{r+1}{n+1}$ such that $(n+1)(f(\frac{r+1}{n+1}) - f(\frac{r}{n+1})) = f'(c_r)$.
c) Show that the sequence $\{B_n'(f;x)\}$ of the derived Bernstein polynomials converges uniformly to f'.