Ark6: Exercises for MAT2400 — Sequences of functions

The exercises on this sheet cover the sections *4.1* to *4.4* Tom's notes. They are ment for the groups on Thursday, Mars 8 and Friday, Mars 10. With the following distribution:

Thursday, Mars 8: No 4, 5, 6, 10, 11, 13, 14
The rest for Friday,

Key words: lim sup and lim inf, Weierstrass' M-test, Derivation and Integation of sequences, Power series, Abel's theorem..

lim sup and lim inf

PROBLEM 1. *(Tom's notes 4.1, Problem 1,2 og 3 (page 85)).* For each of the following sequences $\{a_n\}$ determine $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$:

a) $a_n = (-1)^n$.

b) $a_n = \cos\frac{n\pi}{2}$.

c) $a_n = \arctan(n)\sin(\frac{n\pi}{2})$.

PROBLEM 2. Let $a_n = \cos\sqrt{t + n^2\pi^2}$ for a fixed value of $t > 0$. Find $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$.

PROBLEM 3. Determine $\limsup_{n\to\infty} \frac{1+(-1)^n 2n}{1+3n}$ and $\liminf_{n\to\infty} \frac{1+(-1)^n 2n}{1+3n}$.

PROBLEM 4. Let $\{a_n\}$ be a sequence. Show that there is a subsequence $\{a_{n_k}\}$ and a subsequence $\{a_{m_k}\}$ with $\lim_{k\to\infty} a_{n_k} = \limsup_{n\to\infty} a_n$ and $\lim_{k\to\infty} a_{m_k} = \liminf_{n\to\infty} a_n$.

What can you say about the original sequence if the two subsequences are the same?

PROBLEM 5.

a) Show that

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

and that

$$\liminf_{n\to\infty}(a_n + b_n) \geq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n.$$

Give examples that euqality does not hold.

b) State and prove similar results for the product $\{a_n b_n\}$ of two squences with *positive* terms.

PROBLEM 6. *(Tom's notes 4.1, Problem 6 (page 85)).* Assume that the sequence $\{a_n\}$ is nonnegative and converges to a, and that $b = \limsup_{n\to\infty} b_n$ is finite. Show that $\limsup_{n\to\infty} a_n b_n = ab$. What happens if the terms of the sequence $\{a_n\}$ are negative?

PROBLEM 7. Determine $\limsup_{x\to 0} \frac{x+a}{x+b} \sin^2 \frac{1}{x}$ where $a, b \in \mathbb{R}$.

Weierstass' M-test, derivation and integration of sequences

PROBLEM 8. Show that $\sum_{n=0}^{\infty} \cos nx/(n^2+1)$ converges uniformly on $\mathbb{R}$.

PROBLEM 9.

a) Let $\{a_n\}$ be a sequence and assume that there is a number $s > 1$ such that $|n^s a_n|$ is bounded. Show that $\sum_{n=0}^{\infty} a_n \cos nx$ is uniformly convergent for $x \in \mathbb{R}$ to a continuous function $f(x)$.

b) If we in 9.a) assume that $s > 2$, show that $f(x)$ is diffrentiable and that $\sum_{n=0}^{\infty} n a_n \sin nx$ converges to $-f'(x)$.

c) Let k be an integer and assume that the number s in 9.a) satisfies $s > k+1$. Show that the function $f(x)$ from 9.a) is k times differentiable.

PROBLEM 10. *(Basically Tom's notes 4.2, Problem 6 (page 91)).*

a) Show that $\sum_{n=1}^{\infty} \frac{\sin \frac{x}{n}}{n}$ converges to a continuous function $f(x)$ for all x, and that the convergence is uniform on all closed intervals $[a, b]$.

b) Show that $\sum_{n=1}^{\infty} \frac{\cos \frac{x}{n}}{n^2}$ converges uniformly to $f'(x)$

c) What can you say about the series $\sum_{n=1}^{\infty} (1 - \cos \frac{x}{n})$?

PROBLEM 11. *(Tom's notes 4.2, Problem 8 (page 91)).*

a) Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^x}$ converges uniformly on all intervals $[a, \infty)$ as long as $a > 1$.

b) We let $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$ for $x > 1$. Show that $f'(x) = -\sum_{n=1}^{\infty} \frac{\log n}{n^x}$.

c) Show that $f(x)$ is k times differentiable for any integer k and that $f^{(k)}(x) = \sum_{n=1}^{\infty} (-1)^k \frac{(\log n)^k}{n^x}$.

Power series

PROBLEM 12. For any $r > 0$, find a power series with radius of convergence equal to r.

Find one with radius of convergence equal to 0.

Problem 13.
a) Show that for any polynomial P, we have $\lim_{n\to\infty} \sqrt[n]{|P(n)|} = 1$.
b) Find the radius of convergence of $\sum_{n=1}^{\infty} \frac{2^n x^n}{n^3+1}$.
c) Find the radius of convergence of $\sum_{n=1}^{\infty} \frac{n^a+1}{n^b+1} x^n$, where a and b are natural numbers.

Problem 14. Show that $\sum_{n=0}^{\infty} n^n x^n$ has radius of convergence equal to 0. Find the radius of convergence of $\sum_{n=0}^{\infty} n^n x^{n^2}$.

Abels Theorem

Problem 15. *(Tom's notes 4.4, Problem 1 (page 97)).*
a) Explain why $\frac{1}{x+1} = \sum_{n=0}^{\infty} (-1)^n x^n$ for $|x| < 1$.
b) Show that $\log(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$ for $|x| < 1$.
c) Show that $\log 2 = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1}$.

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