Ark9: Exercises for MAT2400 — Fourier series

The exercises on this sheet cover the sections *4.9* to *4.13*. They are intended for the groups on Thursday, April 12 and Friday, March 30 and April 13.

NB: No group on Thursday, March 29!

The distribution is the following: *Friday, March 30 and April 13:* No 1, 2, 3, 4, 7, 9, 10, 12, 13, 14, 15.
The rest for Thursday, April 12.

Key words: Periodic functions, Fourier series, Dini's test, Integration of Fourier series.

Periodic functions, the spaces D and C_P

PROBLEM 1. *(Tom's notes 4.8, Problem 1 (page 122)).* Show that C_P is a closed subset of $C([-\pi,\pi],\mathbb{C})$.

PROBLEM 2. *(Tom's notes 4.8, Problem 2 (page 122)).* In this problem we shall prove some properties of the space D of piecewise continuous functions on $[-\pi,\pi]$.

a) Show that if f, g are functions from D, then $f+g \in D$ and $fg \in D$.

b) Show that D is a vector space.

c) Show that all functions in D are bounded.

d) Show that all functions in D are Riemann integrable on $[-\pi,\pi]$.

e) Show that $\langle f, g \rangle = \frac{1}{2}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx$ is an inner product on D.

PROBLEM 3.

a) Let the function f on $[0,2\pi]$ be given by $f(x) = x - \pi$. Let $\tilde{f}$ be the periodic extension of f to $\mathbb{R}$ with period 2π. Describe what $\tilde{f}$ looks like, when restricted to $[-\pi,\pi]$.

b) If g is defined in $[\pi,3\pi]$ by $g(x) = x-\pi$ and $\tilde{g}$ denotes the 2π-periodic extension of g to $\mathbb{R}$, what does the restriction of $\tilde{g}$ to $[-\pi,\pi]$ look like?

PROBLEM 4. Let $f(x)$ be the function defined on the entire real line $\mathbb{R}$ which is periodic with period 2π and which is equal to x when $x \in (-\pi,\pi)$ and has $f(\pi) = f(-\pi) = 0$.

a) Show that

$$\frac{x}{2} = \sum_{n=1}^{\infty} (-1)^{(n+1)} \frac{\sin nx}{n},$$

whenever $x \in (-\pi, \pi)$. What happens in the endpoints? Show further that

$$\frac{f(x)}{2} = \sum_{n=1}^{\infty} (-1)^{(n+1)} \frac{\sin nx}{n}$$

for any $x \in \mathbb{R}$. Hint: Use Dini's test and the periodicity.
b) Let g be the 2π-periodic function $g(x) = -f(x - \pi)$. Show that

$$g(x) = -f(x-\pi) = \begin{cases} -x + \pi & \text{if } 0 < x \leq \pi \\ -x - \pi & \text{if } -\pi \leq x < 0 \\ 0 & \text{if } x = 0 \end{cases}$$

c) Use problem 5 on Ark8 to verify that we have

$$\frac{g(x)}{2} = \sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

for $x \in (-\pi, \pi)$.
d) Show that

$$\frac{x}{2} = \frac{\pi}{2} - \sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

whenever $x \in (0, 2\pi)$. Explain why this is not in contradiction with **??**.

Riemann-Lebesgue lemma and Dini's test

Problem 5. Let $I = [a, b] \subseteq [-\pi, \pi]$ be an interval. Let χ_I be the characteristic function of I, *i.e.*, the function such that $\chi_I(x) = 1$ if $x \in I$ and $\chi_I(x) = 0$ if $x \notin I$.
a) Show that the Fourier coefficients of χ_I is given by

$$c_n = \frac{i}{2\pi n}(e^{-ibn} - e^{-ian}),$$

and show that $\lim_{n \to \pm\infty} c_n = 0$.
b) Show that

$$\chi_I(x) = (a - b)/2\pi + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \big((\sin bn - \sin an) \cos nx + (\cos an - \cos bn) \sin nx\big)$$

if $x \neq a, b$. What happens if $x = a$ or $x = b$?

Recall that a *step function* on $[-\pi, \pi]$ is a function $s(x)$ which is a finite linear combination of characteristic functions of intervals contained in $[-\pi, \pi]$, *i.e.,* $s(x) = \sum_{1 \leq k \leq r} a_k \chi_{I_k}(x)$ where the I_k's are intervals contained in $[-\pi, \pi]$ for $k = 1, \ldots, r$.

c) If we denote by c_n the n-th Fourier coefficient of a step function s, show that $\lim_{n \to \pm\infty} c_n = 0$.

PROBLEM 6. Let $f(x)$ be a bounded and Riemann integrable function on $[-\pi, \pi]$. Then — almost by definition of the Riemann integral — if $\epsilon > 0$ is given, we can find a step function $s(x)$ such that

$$\int_{-\pi}^{\pi} |f(x) - s(x)| \, dx < \epsilon.$$

Use this to show that $\lim_{n \to \pm\infty} c_n(f) = 0$, where $c_n(f)$ is the n-th Fourier coefficient of f. HINT: **??** can be usefull.

PROBLEM 7. Let $f(x)$ be a periodic function with period 2π which is p times continuously differentiable for all $x \in \mathbb{R}$, and let c_n be its n-th Fourier coefficient. Show that

$$\lim_{n \to \infty} n^p c_n = 0.$$

HINT: Use partial integration and induction on p.

PROBLEM 8. Let $f(x)$ be a bounded, Riemann integrable function on $[-\pi, \pi]$, and let $g(x) = \int_0^x f(t) \, dt$. Show that the two onesided limits $g(x^+)$ and $g(x^-)$ exist for any $x \in (-\pi, \pi)$, and that the Fourier series of g converges to $(g(x^+) + g(x^-))/2$ for all $x \in [-\pi, \pi]$. HINT: Use Dinis' test.

PROBLEM 9.

a) Let $f(x)$ be integrabel over the interval $[-\pi, \pi]$. Assume that there is an open interval $I \subseteq [-\pi, \pi]$ such that $f(x) = 0$ for all $x \in I$.

Show that the Fourier series of f converges to 0 at all $x \in I$.

b) Let now $f(x)$ and $g(x)$ be two integrabel functions on $[-\pi, \pi]$ and assume that there is an open interval $I \subseteq [-\pi, \pi]$ such that $f(x) = g(x)$ for $x \in I$. Show that if $y \in I$ then the Fourier series of f converges at y if and only if the Fourier series of g converges there. In case the series converge, show that their sums are equal.

PROBLEM 10.

a) Let $f(x)$ be a function on $[-\pi, \pi]$ belonging to the space D. Show that if all the Fourier coefficients of f are zero, then f is identical zero. HINT: Parseval's identity may be usefull.

b) Show that if two functions from D have the same Fourier coefficients, they are equal.

PROBLEM 11. Let $h \in (0, \pi)$.

a) Show that

$$h = -\sum_{n=1}^{\infty} \frac{\sin nh}{n} \cos nx = 0, \tag{✣}$$

whenever $x \in (-\pi, \pi)$ and $\pi > |x| > h$. What happens in the points $x = \pm\pi$ and $x = \pm h$? Check the formulas you obtain by giving x these values against earlier formulas (*i.e.*, from problem **??** on this Ark). HINT: Problem 10 on Ark8.

b) Explain why the equation **??** at the first view seems to be a paradox. Then explain why **??**, after a second though, in fact is not a paradox. HINT: Use the formula $2 \sin \alpha \cos \beta = \sin(\alpha + \beta) + \sin(\alpha - \beta)$ and the Fourier series from problem **??**.

Integration of Fourier series

PROBLEM 12.

a) Find the mean value of $f(x) = \frac{x^2}{4}$ over the interval $[-\pi, \pi]$.

b) Show that we have the equality

$$\frac{x^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^2}$$

for $x \in [-\pi, \pi]$. HINT: $\frac{x}{2} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n}$ in $[-\pi, \pi]$.

c) Show that $\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$.

PROBLEM 13.

a) Show that

$$(x^3 - \pi^2 x)/12 = \sum_{n=1}^{\infty} (-1)^n \frac{\sin nx}{n^3}$$

for all $x \in \mathbb{R}$.

b) Show that

$$(x^4 - 2\pi^2 x^2)/48 = -\frac{7\pi^4}{720} + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\cos nx}{n^4}$$

for all $x \in \mathbb{R}$.

c) Determine $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^4}$.

PROBLEM 14. Asume that a trigonometric series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ converges uniformly.
a) Show that the series is its own Fourier series.
b) Why is **??** not an utterly stupid question?

PROBLEM 15.
a) Show that

$$\sin\frac{x}{2} = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin nx}{4n^2-1} \tag{✻}$$

for $x \in (-\pi, \pi)$.
b) Integrate the series **??** to obtain the Fourier series for $\cos\frac{x}{2}$.
c) Differentiate the series **??** to obtain the Fourier series for $\cos\frac{x}{2}$.
d) Check that the two series thus obtained are equal.

Example of a trigonometric series that is not a Fourier series

PROBLEM 16.
a) Show that the trigonometric series

$$\sum_{n=2}^{\infty}\frac{\sin nx}{\log n} \tag{❀}$$

converges for all x. HINT: Use Dirichlet's criterion, problem 1 on Ark7.
b) Compute the series one obtains by termwise integration of the series **??**.
c) Show that the integrated series diverges for $x = 0$.
d) Show that the series in **??** is not the Fourier series of any function in D. HINT: Use proposition 14.13.1 in Tom's notes.
e) Let $f(x)$ be the function defined by **??**, *i.e.*, $f(x) = \sum_{n=2}^{\infty}\frac{\sin nx}{\log n}$. Show that $f(x)$ is continuous if $x \neq 0$. HINT: Use Dirichlet's criterion to see that **??** converges uniformly on intervals of the form $I = (\delta, \pi)$ and $(-\pi, -\delta)$, where $\delta > 0$.

The behavior of $\sin nx$ and $\cos nx$ when $n \to \infty$

PROBLEM 17. The aim of this exercise is two prove the following, which is about *rational approximation* of irrational numbers needed to understand how $\sin nx$ and $\cos nx$ behaves for n big.

Let an irrational number y and N a natural number be given. Then there is a natural number $n \leq N$ and an integer k such that $|ny - k| < \frac{1}{N+1}$.

a) For any integer r between 1 and N let $f(r) = ry - \lfloor ry \rfloor$. [1]. Show that $0 < f(r) < 1$ for any of the r's, and that $f(r) \neq f(r')$ if $r \neq r'$. Hence the set $A = \{f(r) : 1 \leq r \leq N\} \cup \{0, 1\}$ is contained in $[0, 1]$ and has $N + 2$ elements.

HINT: Show that any of the two cases $f(r) = f(r')$ or $f(r) = 0$ implies that y is rational.

b) Divide $I = [0, 1]$ into the $N + 1$ intervals $I_s = [\frac{s-1}{N+1}, \frac{s}{N+1})$ for $s = 1, \ldots, N$ and $I_{N+1} = [\frac{N-1}{N+1}, 1]$. Show that for some s between 1 and $N+1$ at least two of the members of A are contained in I_s. Conclude by showing that $|(r - r')y - (\lfloor ry \rfloor - \lfloor r'y \rfloor)| < \frac{1}{N+1}$.

The technique used in the last part, is sometimes called the "pigeonhole principle", *i.e.*, if you are to place a certain number of "pigeons" in a certain number of "pigeonholes" and you have more pigeons than holes, then at least two pigeons must be placed in the same hole.

PROBLEM 18. This problem deals with the result that if x is not a rational multiple of 2π, then $B_x = \{nx + 2k\pi : n \in \mathbb{Z}, k \in \mathbb{Z}\}$ is dense in $\mathbb{R}$.

a) Show that if $z \in B_x$ then $-z \in B_x$.

b) Use problem **??** to show that for any $\epsilon > 0$ there is a member $z \in B_x$ such that $|z| < \epsilon$. HINT: Take a look at $x/2\pi$.

c) Let $\overline{B_x}$ denote the closure of B_x, show that $\mathbb{R} = \overline{B_x}$. HINT: Let U be the complement of $\overline{B_x}$. If $u \in U$ and $u > 0$, let $\alpha = \inf\{a :$ There is a b such that $u \in (a, b) \subseteq U\}$. Check that $\alpha > 0$, and get a contradiction by **??**.

PROBLEM 19. Let x be a number that is not a rational multiple of 2π.

a) Show that the two sets $\{\sin nx : n \in \mathbb{N}\}$ and $\{\cos nx : n \in \mathbb{N}\}$ are dense in $[-1, 1]$.

b) Show that neither of the limits $\lim_{n\to\infty} \sin nx$ and $\lim_{n\to\infty} \cos nx$ exists. Show that the series $\sum_{n=1}^{\infty} \cos nx$ and $\sum_{n=1}^{\infty} \sin nx$ both diverge.

c) Let $\sum_{n=0}^{\infty} b_n \sin nx$ be a trigonometric series, and assume that all the coefficients b_n are positive. Show that the series has infinitly many positive and infinitely many negative terms. HINT: Use **??**.

Versjon: Monday, March 26, 2012 12:08:28 PM

[1] If z is any real number, $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z; *e.g.*, $\lfloor 3.14159 \rfloor = 3$ and $\lfloor -2.71828 \rfloor = -3$. We always have $0 \leq z - \lfloor z \rfloor < 1$