MAT2400 Assignment 2 - Solutions

Notation: For any function f of one real variable, $f(a^+)$ denotes the limit of $f(x)$ when x tends to a from above (if it exists); *i.e.*, $f(a^+) = \lim_{t \to a^+} f(t)$. Similarly, $f(a^-)$ denotes the limit of $f(x)$ when x tends to a from below (if it exists).

Problem 1.

The aim of this problem is to study a phenomenon which is called *Gibb's* phenomenon. At every simple jump discontinuity of a function f, the partial sums of the Fourier series of f "overshoots" near the singularity by an amount about 9% of the "jump" of the function.

To be presise, assume that $f(x)$ has a jump singularity at a; *i.e.*, $d = f(a^+) - f(a^-) \neq 0$ and is continuous elsewhere in a neighbourhood of a. For simplicity we assume that $d > 0$. We let $s_n(x)$ be the n-th partial sum of the Fourier series of f. Then there is a sequence $\{x_n\}$ tending to a from above such that $s_n(x_n) > f(a^+) + \alpha d$, where the constant α satisfies $\alpha \approx 0.089$, *i.e.*, about 9%. There is a similar sequence $\{y_n\}$ tending to a from below with $s_n(y_n) < f(a^-) - \alpha d$

In this this problem we will study Gibbs phenomenon for the particular function given in $(-\pi, \pi)$ by:

$$d(x) = \begin{cases} \pi/2 & \text{if } 0 < x < \pi \\ 0 & \text{if } x = 0 \\ -\pi/2 & \text{if } -\pi < x < 0 \end{cases}$$

a) Compute the Fourier coefficients of d, and show that we have the equality

$$d(x) = 2\sum_{k=1}^{\infty} \frac{\sin{(2k-1)x}}{2k-1}$$

for all $x \in (-\pi, \pi)$.

Solution: The function is odd, so its Fourier series is a pure sine-series, and we need only compute

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} d(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi}\frac{\pi}{2}\sin nx\,dx = \Big|_0^{\pi}\frac{1}{n}(-\cos nx) = \frac{(1-(-1)^n)}{n},$$

which equals 0 if n is even and $\frac{2}{n}$ if n is odd. This gives that the Fourier series of $d(x)$ is

$$2\sum_{k=1}^{\infty}\frac{\sin{(2k-1)x}}{2k-1}.$$

Clearly the function $d(x)$ has one-sided derivatives everywhere, hence by Dini's test (or one of the corollaries, **Corollary 14.12.4** in Tom's) the Fourier series converges to $(d(x^+)+d(x^-))/2$ for every x, but this equals $d(x)$ for all x.

b) Let the partial sums of the Fourier series of $d(x)$ be denoted by $d_n(x)$. Show that we have

$$d_n(x) = 2\sum_{k=1}^{n}\frac{\sin{(2k-1)x}}{2k-1} = \int_0^x \frac{\sin 2nt}{\sin t}\,dt.$$

Hint: Compute the derivative of $d_n(x)$ and use that $2\sum_{k=1}^{n}\cos(2k-1)x = \frac{\sin 2nx}{\sin x}$. To prove the last formula, use the for us now well used and classical formula $2\sin\alpha\cos\beta = \sin(\beta+\alpha)-\sin(\beta-\alpha)$.

Solution:

$$2\sum_{k=1}^{n}\cos(2k-1)x = \frac{1}{\sin x}\sum_{k=1}^{n}(\sin 2kx - \sin 2(k-1)x) = \frac{\sin 2nx}{\sin x}$$

for $x\neq 0,\pi$ or $-\pi$ (use the formula in the hint repeatedly with $\beta=(2k-1)x$ and $\alpha = x$), and, in fact, if we interpret the right side as the appropriate limit $\lim \frac{\sin 2nx}{\sin x}$, it holds as well for $x=\pm\pi$ (both sides are zero) and for 0 (both sides are $2n$). Computing the derivative of $d_n(x)$ term by term, we get

$$d_n'(x) = 2\sum_{k=1}^{n}\cos(2k-1)x,$$

and integrating, we obtain

$$d_n(x) = \int_0^x \frac{\sin 2nx}{\sin x}.$$

c) Show that for $t \geq 0$ the following inequality holds true

$$0 \leq t - \sin t \leq t^3/6.$$

Use that inequality to prove that

$$\left| \frac{1}{\sin t} - \frac{1}{t} \right| \leq \frac{\pi}{12} t,$$

when $0 < t \leq \pi/2$.

SOLUTION: It is classical that $\sin t \leq t$ for all $t \geq 0$. To show the other inequality we let

$$f(x) = t - \sin t - t^3/3!$$

and compute $f'(t) = 1 - \cos t - t^2/2$ and $f''(t) = \sin t - t$ which is negative for $t > 0$. Hence $f'(t) < 0$ for $t > 0$ since $f'(0) = 0$. It follows that $f(t) < 0$ for $t > 0$ since $f(0) = 0$. We know that $\frac{2t}{\pi} \leq \sin t$ for $0 \leq t\pi/2$, so we get

$$\left| \frac{1}{\sin t} - \frac{1}{t} \right| = \left| \frac{t - \sin t}{t \sin t} \right| \leq \frac{\pi}{2t^2} \cdot t^3/6 = \frac{\pi}{12} t.$$

d) Prove that for all n and all $0 < x < \pi/2$:

$$\left| d_n(x) - \int_0^{2nx} \frac{\sin u}{u} \, du \right| < \frac{\pi}{24} x^2$$

and use this to prove that for a given $\epsilon > 0$ there is an n_0 such that if $n \geq n_0$, then

$$d_n(\pi/2n) > \pi/2 + \alpha\pi - \epsilon$$

where the constant α is given by $\alpha = \pi^{-1}(\int_0^{\pi} \frac{\sin u}{u} \, du - \pi/2)$. Hence

$$d_n(\pi/2n) \geq \pi/2 + 0.089\pi.$$

because one may compute $\alpha = 0.08949\ldots$. (You can consider that value as given!).

Solution: Integrating the inequality in d), we get

$$\left|\int_0^x \frac{\sin 2nt}{\sin t}\,dt - \int_0^x \frac{\sin 2nt}{t}\,dt\right| \le \int_0^x \frac{\pi t}{12} = \frac{\pi}{24}x^2.$$

Substituting $u = 2nt$ in the second integral and using xxx, we get

$$\left|d_n(x) - \int_0^{2nx} \frac{\sin u}{u}\,du\right| \le \frac{\pi}{24}x^2.$$

Now, we put $x = \pi/2n$ in the formula above to get

$$d_n(\pi/2n) \ge \int_0^{\pi} \frac{\sin u}{u}\,du - \frac{\pi^3}{96}n^{-2} > \pi/2 + \alpha\pi - \epsilon$$

once n is so big that $\frac{\pi^3}{96}n^{-2} < \epsilon$.

Problem 2.

Let $C = C([0,1],\mathbb{R})$ be the Banach space of continuous real valued functions on the interval $[0,1]$ with norm given by $\|f\| = \sup\{|f(x)| : x \in [0,1]\}$. Fix an element $g \in C$, and let $I\colon C \to C$ be the map given by

$$I(f)(x) = \int_0^x f(t)g(t)\,dt.$$

a) Show that I is a bounded linear map; that is, I is linear and there is a positive constant M such that $\|I(f)\| \le M\,\|f\|$ for all $f \in C$. Determine the least such constant if g is a positive function.

SOLUTION: I is linear by wellknown properties of the integral (in fact linearity). To see that I is bounded, we compute

$$\begin{aligned} |I(f)(x)| = \left| \int_0^x f(t)g(t)\,dt \right| &\leq \int_0^x |f(t)g(t)|\,dt \leq \\ &\leq \int_0^1 |f(t)g(t)|\,dt \leq \sup |f(t)g(t)| \\ &\leq \sup |f(t)| \sup |g(t)| = \|f\|\,M, \end{aligned}$$

where $M = \sup |g(t)|$. Hence $\|I(f)\| \leq \|f\|\,M$. If the function g is positive, we compute $\|I(1)\| = \sup |g(t)|$. Hence $M = \sup |g(t)|$ is the smallest constant we can use.

b) Show that the map $I\colon C \to C$ is uniformly continuous.
SOLUTION: Let $\epsilon > 0$ be given, and let the corresponding $\delta > 0$ be $\delta = \epsilon/M$. Then

$$\|I(f) - I(g)\| \leq M\,\|f - g\| < M \cdot \epsilon/M = \epsilon$$

whenever $\|f - g\| < \delta$,

c) Show that for any bounded subset $A \subseteq C$ the set $I(A) \subseteq C$ is equicontinuous.
SOLUTION: Let K be a bound for A, that is $\|f\| \leq K$ for all $f \in A$. We have

$$|I(f)(x) - I(f)(y)| = \left| \int_y^x f(t)g(t)\,dt \right| \leq \int_y^x |f(t)g(t)|\,dt \leq |x - y|\,KM$$

for $f \in A$. Then, given $\epsilon > 0$, we put $\delta = \epsilon/KM$, and obtain

$$|I(f)(x) - I(f)(y)| \leq |x - y|\,KM \leq \epsilon/KM \cdot KM = \epsilon$$

once $|x - y| < \delta$, and this holds for all $f \in A$.

d) Show that the closure $\overline{I(A)}$ is a compact subset.

SOLUTION: We want to apply the Arzela-Ascoli theorem. Now $\overline{I(A)}$ is equicontinuous since $I(A)$ is; indeed, if $\epsilon > 0$ is given, choose $\delta > 0$ such that $|I(g)(x) - I(g)(y)| < \epsilon/3$ for all $g \in A$ and for all $|x - y| < \delta$. Pick an element $F \in \overline{I(A)}$ and let $I(f_n)$ be a sequence converging (uniformly) to F. We have

$$|F(x) - F(y)| \leq |F(x) - I(f_n)(x)| + |I(f_n)(x) - I(f_n)(y)| + |I(f_n)(y) - F(y)|$$

Let $\epsilon > 0$ be given. Choose N such that $n > N$ gives $|F(x) - I(f_n)(x)| < \epsilon/3$ for all x. Then we get by the above inequality.

$$|F(x) - F(y)| < \epsilon.$$

Too see that $\overline{I(A)}$ is bounded, use that the norm is continuous, hence if $I(f_n)$ converges to F, then $\|F\| = \lim_{n\to\infty} \|f_n\| < KM$. It follows from the A&A theorem, that $\overline{I(A)}$ is compact. (It is closed by definition).

e) For each real number $\lambda \neq 0$, let $V_\lambda = \{f \in C : I(f) = \lambda f\}$. Show that V_λ is a subvector space of C. Determine all functions in V_λ.

SOLUTION: It is clear that V_λ is a sub vector space (closed under addition and scalar multiplication). An element f lies in V_λ if $\int_0^x f(t)g(t)\,dt = \lambda f$. The left side of this equation is differentiable (integrals of continuous functions are) hence f is differentiable, and $\lambda f' = fg$. This is a first order differential equation with solution $f(x) = Ce^{\frac{1}{\lambda}\int_0^x g(t)\,dt}$ if $\lambda \neq 0$, but since $\lambda f(x) = \int_0^x f(t)g(t)\,dt$, we see that $f(0) = 0$, hence $C = 0$, and $f \equiv 0$; meaning that $V_\lambda = 0$. If $\lambda = 0$, it is a little more complicated. Then we get $f(x)g(x) \equiv 0$, hence V_0 is the subspace $\{f : f(x)g(x) \equiv 0\}$; and if *e.g.*, g is positive, we get $f \equiv 0$.

Problem 3.

Let $F(x)$ be a strictly increasing function. For any half open interval $I = (a, b]$ define $m(I) = F(b) - F(a)$, and for any set $E \subseteq \mathbb{R}$, let

$$\nu^*(E) = \inf\{\sum_{I \in \mathcal{A}} m(I) : \mathcal{A}\}$$

where $\mathcal{A}$ runs through all countable coverings of E by half open intervals $(a, b]$.

a) Show that $\nu^*(E) \geq 0$, and that ν^* is monotone; *i.e.*, $\nu^*(E') \leq \nu^*(E)$ whenever $E' \subseteq E$.

Solution: Since F is increasing, $m(I) = F(b) - F(a) > 0$. Hence $\nu^*(E) \geq 0$, $\nu^*(E)$ being the supremum of a set of positive numbers. If $E' \subseteq E$, then any covering of E (of the type we use) is also a covering of E' (of the type we use). Hence $\nu^*(E')$ is the supremum of a smaller set than $\nu^*(E)$, so $\nu^*(E') \leq \nu^*(E)$.

b) Show that ν^* is semiadditive; that is

$$\nu^*(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \nu^*(E_n)$$

for any family $\{E_n\}$ of subsets of $\mathbb{R}$.
Solution: This is word by word the same proof as of **Proposition 5.1.4** page 146 in Tom's notes. Take a look at that.

c) If $x \in \mathbb{R}$, show that $\nu^*(\{x\}) = F(x) - F(x^-)$, and hence $\nu^*\{x\} = 0$ if and only if F is continuous from the left at x.
Solution: The sequence $F(x - 1/n)$, where $n \in \mathbb{N}$, is increasing with $F(x^-)$ as limit, hence $F(x-1/n) \leq F(x^-)$ for all n. Any half open interval $(a, b]$ containing x contains an interval of the form $(x - 1/n, x]$ where $n \in \mathbb{N}$. Hence

$$m(I) = F(b) - F(a) \geq F(x) - F(x - 1/n) \geq F(x) - F(x^-)$$

This shows that $\nu^*(\{x\}) \geq F(x) - F(x^-)$. On the other hand, $\nu^*(\{x\}) \leq m((x - 1/n, x]) = F(x) - F(x - 1/n)$ for all n, hence $\nu^*(\{x\}) \leq \inf_{n\in\mathbb{N}}\{F(x) - F(x - 1/n)\} = F(x) - F(x^-)$; and thus $\nu^*(\{x\}) = F(x) - F(x^-)$. The function F is continuous from the left at x if and only if $F(x^-) = F(x)$, hence if and only if $\nu^*\{x\} = 0$, by what we just saw.