4.10.6

a) By the Binomial Formula:

$$1 = 1^N = (x+(1-x))^N = \sum_{k=0}^{N} \binom{N}{k} x^k (1-x)^{N-k},$$

as wanted.

b) Have $\frac{k}{N}\binom{N}{k} = \binom{N-1}{k-1}$ for $k \neq 0$, and 0 for $k=0$

$$\sum_{k=0}^{N} \frac{k}{N}\binom{N}{k} x^k (1-x)^{N-k} = \sum_{k=1}^{N} \binom{N-1}{k-1} x^k (1-x)^{N-k}$$

Let $l = k-1$. Then $\sum_{k=1}^{N} \binom{N-1}{k-1} x^k (1-x)^{N-k} = \sum_{l=0}^{N-1} \binom{N-1}{l} x^{l+1} (1-x)^{N-l-1}$

So $$\sum_{k=0}^{N} \frac{k}{N}\binom{N}{k} x^k (1-x)^{N-k} = x \cdot \sum_{l=0}^{N-1} \binom{N-1}{l} x^l (1-x)^{(N-1)-l}$$

$$= x \cdot 1 \quad \text{by a)}$$

$$= x.$$

c) Have $\frac{k(k-1)}{N^2}\binom{N}{k} = \frac{N-1}{N}\binom{N-2}{k-2}$ for $k \geq 2$, and 0 for $k=0, k=1$.

$$\sum_{k=0}^{N} \left(\frac{k}{N} - x\right)^2 \binom{N}{k} x^k (1-x)^{N-k} = \sum_{k=0}^{N} \frac{1}{N^2}\left(k(k-1) + (1-2xN)k + N^2x^2\right)$$

$$= \sum_{k=0}^{N} \frac{k(k-1)}{N^2}\binom{N}{k} x^k (1-x)^{N-k} + \sum_{k=0}^{N} \frac{1-2xN}{N}\frac{k}{N}\binom{N}{k} x^k (1-x)^{N-k}$$
$$+ \sum_{k=0}^{N} x^2 \binom{N}{k} x^k (1-x)^{N-k}$$

$$= \sum_{k=2}^{N} \frac{N-1}{N}\binom{N-2}{k-2} x^k (1-x)^{N-k} + \frac{1-2xN}{N}x + x^2 \qquad \text{Choose } l = k-2$$

$$= \sum_{l=0}^{N-2} \frac{N-1}{N}\binom{N-2}{l} x^2 x^l (1-x)^{N-2-l} + \frac{x}{N}(1-2xN + xN)$$

4.10.6 c) cont.

$$= \frac{N-1}{N} x^2 + \frac{x}{N}(1-Nx) = \frac{x}{N}(Nx - x + 1 - Nx) = \frac{1}{N} x(1-x)$$

as wanted.

d) $$|f(x) - P_n(x)| = |f(x)\cdot 1 - P_n(x)| = \left|f(x)\sum_{k=0}^{n}\binom{n}{k} x^k (1-x)^{n-k} - P_n(x)\right|$$

$$= \left|\sum_{k=0}^{n} f(x)\binom{n}{k} x^k (1-x)^{n-k} - \sum_{k=0}^{n} f\left(\tfrac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}\right|$$

$$= \left|\sum_{k=0}^{n}\left(f(x) - f\left(\tfrac{k}{n}\right)\right)\binom{n}{k} x^k (1-x)^{n-k}\right|$$

$$\leq \sum_{k=0}^{n}\left|f(x) - f\left(\tfrac{k}{n}\right)\right|\left|\binom{n}{k}\right| x^k (1-x)^{n-k}$$

as $|x| = x$ and $|1-x| = 1-x$.

e) As f is a continuous function on a compact set ($[0,1]$), we have that f is uniformly continuous on $[0,1]$ so there exists a δ s.t. $|f(u) - f(v)| < \frac{\varepsilon}{2}$ whenever $|u-v| < \delta$.

We split the sum from d) into the set of k's s.t. $|\frac{k}{n} - x| < \delta$ and the set of k's s.t. $|\frac{k}{n} - x| \geq \delta$. For the set of k's s.t. $|\frac{k}{n} - x| < \delta$, we have $|f(x) - f(\frac{k}{n})| < \frac{\varepsilon}{2}$, so we have

$$\sum_{\{k : |\frac{k}{n} - x| < \delta\}} \left|f(x) - f\left(\tfrac{k}{n}\right)\right|\binom{n}{k} x^k (1-x)^{n-k} < \frac{\varepsilon}{2} \sum_{\{k : |\frac{k}{n} - x| < \delta\}} \binom{n}{k} x^k (1-x)^{n-k}$$

$$< \frac{\varepsilon}{2} \sum_{k=0}^{n} \binom{n}{k} x^n (1-x)^{n-k} = \frac{\varepsilon}{2} \text{ by a).}$$

as wanted.

f) By the extreme value theorem, f has a maximum in $[0,1]$, and we must therefore have an M s.t. $|f(x)| \leq M$ for all $x \in [0,1]$.

We have

$$\sum_{\{k:|\frac{k}{n}-x|\geq\delta\}} \left|f(x)-f\left(\tfrac{k}{n}\right)\right|\binom{n}{k}x^k(1-x)^{n-k} \leq \sum_{\{k:|\frac{k}{n}-x|\geq\delta\}}\left(|f(x)|+\left|f\left(\tfrac{k}{n}\right)\right|\right)\binom{n}{k}x^k(1-x)^{n-k}$$

$$\leq 2M \sum_{\{k:|\frac{k}{n}-x|\geq\delta\}}\binom{n}{k}x^k(1-x)^{n-k}$$

$$\leq 2M \sum_{k=0}^{n}\left(\frac{\frac{k}{n}-x}{\delta}\right)^2\binom{n}{k}x^k(1-x)^{n-k}$$

where in the last inequality we use that for the k's in the original sum, $\left(\frac{\frac{k}{n}-x}{\delta}\right)^2 \geq 1$, while for the new elements in the sum, we have $\left(\frac{\frac{k}{n}-x}{\delta}\right)^2 \geq 0$.

So we use e) on this to get

$$2M\sum_{k=0}^{n}\frac{\left(\frac{k}{n}-x\right)^2}{\delta^2}\binom{n}{k}x^k(1-x)^{n-k} = \frac{2M}{\delta^2 n}x(1-x) \leq \frac{M}{2\delta^2 n}$$

as $x(1-x) \leq \frac{1}{4}$ when $x \in [0,1]$.

g) We can choose n large enough that $\frac{M}{2\delta^2 n} < \frac{\varepsilon}{2}$, and we can combine e) and f) to get $|f(x)-P_n(x)| \leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$. As this n works for all $x \in [0,1]$ we have that $\sup_{x\in[0,1]}\{|f(x)-P_n(x)|\} < \varepsilon$ when n is large enough.

Therefore, we have that P_n converges uniformly to f.