1.1 ①

$x \cdot y$ is even $\Rightarrow$ (x is even) or (y is even)

Contrapositive:

not((x is even) or (y is even)) $\Rightarrow$ $x \cdot y$ is not even.

If neither x nor y are even, they must both be odd.

(x is odd) and (y is odd) $\Rightarrow$ $x \cdot y$ is odd.

We will prove this statement.

If x is odd, we have $x = 2k+1$

If y is odd, we have $y = 2l+1$

So
$$
\begin{aligned}
x \cdot y = (2k+1)(2l+1) &= 4kl + 2k + 2l + 1 \\
&= 2(2kl + k + l) + 1 \\
&= 2m + 1,
\end{aligned}
$$

which is odd. $\blacksquare$

(2)

$x+y$ is even $\Rightarrow$ (x and y are even) or (x and y are odd)

Split into two cases:

Assume: $x+y$ is even

and Case 1: x is even.

Then $x=2k$, $x+y=2l$

So $y=x+y-x=2l-2k=2(l-k)$,

which is even.

Case 2: x is odd.

Then $x=2k+1$, $x+y=2l$

So $y=x+y-x=2l-(2k+1)$

$=2l-2k-1=2(l-k)-1$

$=2(l-k-1)+1$

which is odd.

As x has to be either even or odd, these are all the possible cases, and either both are odd or both even. ■

Alternate proof: The contrapositive is

one of x, y is even, the other odd $\Rightarrow$ $x+y$ is odd.

Assume x is even, then y is odd.

Then, $x=2k$, $y=2l+1$, $x+y=2(k+l)+1$, which is odd.

If x is odd, then y must be even, so

$x=2k+1$, $y=2l$, $x+y=2(k+l)+1$, which is odd.

■

(3) n^2 divisible by 3 $\Rightarrow$ n divisible by 3.

Contrapositive:

n not divisible by 3 $\Rightarrow$ n^2 not divisible by 3.

If n is not divisible by 3, then $n = 3 \cdot k + l$, where l is either 1 or 2.

Then $n^2 = (3k+l)^2 = 9k^2 + 6k + l^2 = 3(3k^2+2k) + l^2$.

If $l=1$, then $l^2=1$, so n^2 is not divisible by 3.

If $l=2$, then $l^2 = 4 = 3+1$, so $n^2 = 3(3k^2+2k+1) + 1$, and it is not divisible by 3. $\boxed{}$

Let us now show that $\sqrt{3}$ is irrational.

Assume $\sqrt{3}$ is rational, derive a contradiction.

Let us assume $\sqrt{3} = \frac{m}{n}$, where m and n have <u>no</u> common factors. Then $3 = \frac{m^2}{n^2}$, so $3n^2 = m^2$.

We have m^2 divisible by 3 $\Rightarrow$ m is divisible by 3, $m = 3k$.

Then $3n^2 = m^2 = (3k)^2 = 9k^2$, so $n^2 = 3k^2$.

Therefore, n^2 is divisible by 3 $\Rightarrow$ n is divisible by 3.

So both m and n are divisible by 3, contradicting the (reasonable) assumption that they had no common factors. Therefore, the less reasonable assumption, $\sqrt{3}$ is rational, must be false.

$\lightning$

④

ⓐ Let $r=\frac{a}{b}$, $s=\frac{c}{d}$, $b\neq 0$, $d\neq 0$, a,b,c,d integers.

Then:

$$r+s=\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}, \quad ad+bc,\ bd \text{ integers}, \ bd\neq 0.$$

$$r-s=\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}, \quad ad-bc,\ bd, \text{ integers}, \ bd\neq 0$$

$$r\cdot s=\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}, \quad ac,\ bd \text{ integers}, \ bd\neq 0$$

$$\frac{r}{s}=\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{ad}{b\cdot c}, \quad ad,\ bc \text{ integers}, \ bc\neq 0 \text{ providing } c\neq 0 \text{ i.e. } s\neq 0.$$

So all these are rational numbers.

ⓑ If $r+a=s$, with s rational, then $a=s-r$. But $s-r$ is then rational, while a was irrational, a contradiction. So s must be irrational.

If $r-a=s$, with s rational, then $a=r-s$. But $r-s$ is then rational, while a was irrational, a contradiction. So s must be irrational.

If $r\cdot a=s$, with s rational and $r\neq 0$, then $a=\frac{s}{r}$. But $\frac{s}{r}$ is rational, while a was irrational, a contradiction. So s must be irrational.

④ cont.

If $\frac{r}{a} = s$, with s rational, $r \neq 0$, then

$a = \frac{r}{s}$. But $\frac{r}{s}$ is rational ($s \neq 0$ as $r \neq 0$), while a ~~was~~ is irrational, a contradiction. So s must be irrational.

If $\frac{a}{r} = s$, with s rational, $r \neq 0$, then

$a = rs$. But rs is rational, while a was irrational, a contradiction. So s must be irrational.

ⓒ $a+b$: $a = \sqrt{2}$, $b = -\sqrt{2}$ $\quad a+b = \sqrt{2} + (-\sqrt{2}) = 0$ rational

$a = \frac{\pi}{2}$, $b = \frac{\pi}{2}$ $\quad a+b = \frac{\pi}{2} + \frac{\pi}{2} = \pi$ irrational

ab: $a = \sqrt{2}$, $b = \sqrt{2}$ $\quad ab = \sqrt{2} \cdot \sqrt{2} = 2$ rational

$a = \sqrt[4]{2}$, $b = \sqrt[4]{2}$ $\quad ab = \sqrt[4]{2} \cdot \sqrt[4]{2} = \sqrt{2}$ irrational

1.21

1

Will show $[0,2] \cup [1,3] \subseteq [0,3]$ and $[0,3] \subseteq [0,2] \cup [1,3]$

Let $x \in [0,2] \cup [1,3]$. Then $x \in [0,2]$ or $x \in [1,3]$.
If $x \in [0,2]$, then $x \in [0,3]$, and if $x \in [1,3]$ then $x \in [0,3]$.
So $x \in [0,2]$ or $x \in [1,3]$ $\Rightarrow x \in [0,3]$. I.e. $[0,2] \cup [1,3] \subseteq [0,3]$.

Let $x \in [0,3]$. If $x \leq 2$, then $x \in [0,2]$. Otherwise, $x \in (2,3]$.
And if $x \in (2,3]$, then $x \in [1,3]$. So $x \in [0,2]$ or $x \in [1,3]$, i.e.
$x \in [0,2] \cup [1,3]$. So $[0,3] \subseteq [0,2] \cup [1,3]$.

We have $[0,2] \cup [1,3] \subseteq [0,3]$ and $[0,3] \subseteq [0,2] \cup [1,3]$, so

$$[0,2] \cup [1,3] = [0,3].$$

$\blacksquare$

Will show $[0,2] \cap [1,3] \subseteq [1,2]$ and $[1,2] \subseteq [0,2] \cap [1,3]$.

Let $x \in [0,2] \cap [1,3]$. Then $x \in [0,2]$ and $x \in [1,3]$.
As $x \in [0,2]$, we have $x \leq 2$, and as $x \in [1,3]$ we have $x \geq 1$.
So $x \in [1,3]$, i.e. $[0,2] \cap [1,3] \subseteq [1,2]$.

Let $x \in [1,2]$. As $[1,2] \subseteq [0,2]$ and $[1,2] \subseteq [1,3]$, we have
$x \in [0,2]$ and $x \in [1,3]$, so $x \in [0,2] \cap [1,3]$, i.e. $[1,2] \subseteq [0,2] \cap [1,3]$.

We have $[0,2] \cap [1,3] \subseteq [1,2]$ and $[1,2] \subseteq [0,2] \cap [1,3]$, so

$$[0,2] \cap [1,3] = [1,2].$$

$\blacksquare$

(2) Will show $(-\infty,0)^c \subseteq [0,\infty)$ and $[0,\infty) \subseteq (-\infty,0)^c$.

Let $x \in (-\infty,0)^c$, i.e. $x \notin (-\infty,0)$. Then $x \not< 0 \Leftrightarrow x \geq 0$, so $x \in [0,\infty)$, i.e. $(-\infty,0)^c \subseteq [0,\infty)$.

Let $x \in [0,\infty)$, i.e. $x \geq 0$. Then $x \not< 0$, so $x \notin (-\infty,0)$, i.e. $x \in (-\infty,0)^c$. So $[0,\infty) \subseteq (-\infty,0)^c$.

We have $(-\infty,0)^c \subseteq [0,\infty)$ and $[0,\infty) \subseteq (-\infty,0)^c$, so $[0,\infty) = (-\infty,0)^c$. ∎

(5) Will show $B \cup (A_1 \cap A_2 \cap \dots \cap A_n) \subseteq (B \cup A_1) \cap (B \cup A_2) \cap \dots \cap (B \cup A_n)$ and $(B \cup A_1) \cap (B \cup A_2) \cap \dots (B \cup A_n) \subseteq B \cup (A_1 \cap A_2 \cap \dots A_n)$.

Let $x \in B \cup (A_1 \cap A_2 \cap \dots \cap A_n)$. Then $x \in B$ or $x \in A_i$ for all $i \leq n$. If $x \in B$ then $x \in B \cup A_i$ for all $i \leq n$ (($x \in B$ or $x \in A_i$) is true). On the other hand, if $x \in A_i$ for all $i \leq n$, then $x \in B \cup A_i$ for all $i \leq n$ as well. Either way, we have $x \in (B \cup A_1) \cap (B \cup A_2) \cap \dots \cap (B \cup A_n)$, so
$$B \cup (A_1 \cap A_2 \cap \dots \cap A_n) \subseteq (B \cup A_1) \cap (B \cup A_2) \cap \dots \cap (B \cup A_n)$$

Let $x \in (B \cup A_1) \cap (B \cup A_2) \cap \dots \cap (B \cup A_n)$. Then $x \in (B \cup A_i)$ for all $i \leq n$. Either $x \in A_i$ for all $i \leq n$, or there exists a $j \leq n$ with $x \notin A_j$. But as $x \in B \cup A_j$, we must then have $x \in B$. So we have $x \in B$ or $x \in A_i$ for all $i \leq n$, i.e. either $x \in B$ or $x \in A_1 \cap A_2 \cap \dots \cap A_n$. Therefore, we have $x \in B \cup (A_1 \cap A_2 \cap \dots A_n)$. This proves
$$(B \cup A_1) \cap (B \cup A_2) \cap \dots \cap (B \cup A_n) \subseteq B \cup (A_1 \cap A_2 \cap \dots \cap A_n).$$

We have $B \cup (A_1 \cap A_2 \cap \dots \cap A_n) \subseteq (B \cup A_1) \cap (B \cup A_2) \cap \dots \cap (B \cup A_n)$ and $(B \cup A_1) \cap (B \cup A_2) \cap \dots (B \cup A_n) \subseteq B \cup (A_1 \cap A_2 \cap \dots A_n)$, so
$$B \cup (A_1 \cap A_2 \cap \dots \cap A_n) = (B \cup A_1) \cap (B \cup A_2) \cap \dots \cap (B \cup A_n).$$

∎

⑥ Will show $(A_1 \cap A_2 \cap \dots \cap A_n)^c \in A_1^c \cup A_2^c \cup \dots \cup A_n^c$ and

$A_1^c \cup A_2^c \cup \dots \cup A_n^c \subseteq (A_1 \cap A_2 \cap \dots \cap A_n)^c$.

Let $x \in (A_1 \cap A_2 \cap \dots \cap A_n)^c$, i.e. $x \notin A_1 \cap A_2 \cap \dots A_n$.

As $x \notin A_1 \cap A_2 \cap \dots \cap A_n$, there must exist a $j \leq n$ s.t. $x \notin A_j$, i.e. $x \in A_j^c$. And if $x \in A_j^c$, then $x \in A_1^c \cup A_2^c \cup \dots \cup A_n^c$, so $(A_1 \cap A_2 \cap \dots \cap A_n)^c \subseteq A_1^c \cup A_2^c \cup \dots \cup A_n^c$.

Let $x \in A_1^c \cup A_2^c \cup \dots \cup A_n^c$. Then there exists a $j \leq n$ s.t. $x \in A_j^c$, i.e. $x \notin A_j$. As $x \notin A_j$, we have $x \notin A_1 \cap A_2 \cap \dots \cap A_n$, so $x \in (A_1 \cap A_2 \cap \dots A_n)^c$, i.e. $A_1^c \cup A_2^c \cup \dots \cup A_n^c \subseteq (A_1 \cap A_2 \cap \dots \cap A_n)^c$.

As we have $(A_1 \cap A_2 \cap \dots \cap A_n)^c \subseteq A_1^c \cup A_2^c \cup \dots \cup A_n^c$

and $A_1^c \cup A_2^c \cup \dots \cup A_n^c \in (A_1 \cap A_2 \cap \dots \cap A_n)^c$, we have

$$(A_1 \cap A_2 \cap \dots \cap A_n)^c = A_1^c \cup A_2^c \cup \dots \cup A_n^c \qquad \blacksquare$$

⑧ Will show $(A\cup B)\times C \subseteq (A\times C)\cup(B\times C)$ and $(A\times C)\cup(B\times C)\subseteq(A\cup B)\times C$

Let $(x,y)\in(A\cup B)\times C$, i.e. $x\in A\cup B$, $y\in C$. Then $x\in A$ or $x\in B$.
If $x\in A$, then $(x,y)\in A\times C$. If $x\in B$, then $(x,y)\in B\times C$.
So $(x,y)\in A\times C$ or $(x,y)\in B\times C$, i.e. $(x,y)\in(A\times C)\cup(B\times C)$.
Therefore, $(A\cup B)\times C\subseteq(A\times C)\cup(B\times C)$.

Let $(x,y)\in(A\times C)\cup(B\times C)$, i.e. $(x,y)\in A\times C$ or $(x,y)\in B\times C$.
Then $y\in C$ and ($x\in A$ or $x\in B$). So $x\in A\cup B$. Therefore
$(x,y)\in(A\cup B)\times C$, and we have $(A\times C)\cup(B\times C)\subseteq(A\cup B)\times C$.

As we have $(A\cup B)\times C\subseteq(A\times C)\cup(B\times C)$ and
$(A\times C)\cup(B\times C)\subseteq(A\cup B)\times C$, we have

$$(A\cup B)\times C=(A\times C)\cup(B\times C).$$

$\blacksquare$

Will show $(A\cap B)\times C\subseteq(A\times C)\cap(B\times C)$ and $(A\times C)\cap(B\times C)\subseteq(A\cap B)\times C$.

Let $(x,y)\in(A\cap B)\times C$. Then $x\in A\cap B$ and $y\in C$. So
$x\in A$ and $x\in B$. As $x\in A$, $(x,y)\in A\times C$, and as $x\in B$, $(x,y)\in B\times C$.
So $x\in(A\times C)\cap(B\times C)$, and we have $(A\cap B)\times C\subseteq(A\times C)\cap(B\times C)$.

Let $(x,y)\in(A\times C)\cap(B\times C)$. Then $(x,y)\in A\times C$ and $(x,y)\in B\times C$.
So $y\in C$, $x\in A$ and $x\in B$, giving us $x\in A\cap B$. We have
$(x,y)\in(A\cap B)\times C$, giving us $(A\times C)\cap(B\times C)\subseteq(A\cap B)\times C$.

As we have both $(A\cap B)\times C\subseteq(A\times C)\cap(B\times C)$ and
$(A\times C)\cap(B\times C)\subseteq(A\cap B)\times C$, we have

$$(A\cap B)\times C=(A\times C)\cap(B\times C).$$

$\blacksquare$

1.31

(1)

We have $\bigcup_{n\in\mathbb{N}}[-n,n] \subseteq \mathbb{R}$, as our universe is $\mathbb{R}$, everything is in $\mathbb{R}$. Will show $\mathbb{R} \subseteq \bigcup_{n\in\mathbb{N}}[-n,n]$.

Let $x\in\mathbb{R}$. Let $m=\lceil |x| \rceil$, where $\lceil \cdot \rceil$ is the "ceiling"-function, (i.e. $\lceil |x| \rceil$ is the smallest natural number greater than or equal to the absolute value of x). Then $x\in[-m,m]$, as $m\geq |x| \geq x$ and $-m \leq -|x| \leq x$.

Since there exist an $m\in\mathbb{N}$ with $x\in[-m,m]$, we have $x\in \bigcup_{n\in\mathbb{N}}[-n,n]$, so $\mathbb{R} \subset \bigcup_{n\in\mathbb{N}}[-n,n]$.

(2)

As $0\in\left(-\frac{1}{n},\frac{1}{n}\right)$ for all $n\in\mathbb{N}$, we have $0\in\bigcap_{n\in\mathbb{N}}\left(-\frac{1}{n},\frac{1}{n}\right)$, i.e. $\{0\}\subseteq\bigcap_{n\in\mathbb{N}}\left(-\frac{1}{n},\frac{1}{n}\right)$.

So if $x\in\bigcap_{n\in\mathbb{N}}\left(-\frac{1}{n},\frac{1}{n}\right)$, must show $x=0$.

Contrapositive: If $x\neq 0$, then $x\notin\bigcap_{n\in\mathbb{N}}\left(-\frac{1}{n},\frac{1}{n}\right)$.

Assume $x\neq 0$, and let $m=\left\lceil\frac{1}{|x|}\right\rceil$. Then $m\geq\frac{1}{|x|} \Rightarrow |x|>\frac{1}{m}$.

So $x\notin\left(-\frac{1}{m},\frac{1}{m}\right)$, implying $x\in\bigcap_{n\in\mathbb{N}}\left(-\frac{1}{n},\frac{1}{n}\right)$, which is what we needed.

(3) Must show $\bigcup_{n\in\mathbb{N}} [\frac{1}{n},1] \subseteq (0,1]$ and $(0,1] \subseteq \bigcup_{n\in\mathbb{N}} [\frac{1}{n},1]$.

If $x \in \bigcup_{n\in\mathbb{N}} [\frac{1}{n},1]$, there exists an $m\in\mathbb{N}$ s.t. $x \in [\frac{1}{m},1] \subseteq (0,1]$, so $x\in(0,1]$.

If $x\in(0,1]$, let $m = \lceil \frac{1}{x} \rceil$. Then $m \geq \frac{1}{x} \Rightarrow x \geq \frac{1}{m}$.

So $x \in [\frac{1}{m},1]$, for an $m\in\mathbb{N}$. $\Rightarrow x \in \bigcup_{n\in\mathbb{N}} [\frac{1}{n},1]$, which is what we needed. $\square$

(4) We have $\emptyset \subseteq A$ for any set A, so we only need to prove

$\bigcap_{n\in\mathbb{N}} (0,\frac{1}{n}] \subseteq \emptyset$, i.e. $x\in\mathbb{R} \Rightarrow x \notin \bigcap_{n\in\mathbb{N}} (0,\frac{1}{n}]$.

If $x \leq 0$, we see $x \notin (0,\frac{1}{n}]$ for any n, so $x \in \bigcap_{n\in\mathbb{N}} (0,\frac{1}{n}]$.

For $x>0$, choose $m = \lceil \frac{1}{x} \rceil + 1$. Then $m > \frac{1}{x} \Rightarrow x > \frac{1}{m}$.

So $x \notin (0,\frac{1}{m}] \Rightarrow x \notin \bigcap_{n\in\mathbb{N}} (0,\frac{1}{n}]$, which is what we needed. $\square$

(5) Will show $B\cap\left(\bigcup_{A\in\mathcal{A}} A\right) \subseteq \bigcup_{A\in\mathcal{A}} (B\cap A)$ and $\bigcup_{A\in\mathcal{A}}(B\cap A) \subseteq B\cap\left(\bigcup_{A\in\mathcal{A}} A\right)$

Let $x \in B\cap\left(\bigcup_{A\in\mathcal{A}} A\right)$. Then $x\in B$ and there exist $A'\in\mathcal{A}$ with $x\in A'$.

Then $x \in B\cap A'$, so we have $x \in \bigcup_{A\in\mathcal{A}}(B\cap A)$.

Let $x\in \bigcup_{A\in\mathcal{A}}(B\cap A)$. Then there exists an $A'\in\mathcal{A}$ s.t. $x\in B\cap A'$, i.e,

$x\in B$ and $x \in A'$. So $x \in B\cap\left(\bigcup_{A\in\mathcal{A}} A\right)$. $\square$

Will show $B\cup\left(\bigcap_{A\in\mathcal{A}} A\right) \subseteq \bigcap_{A\in\mathcal{A}}(B\cup A)$ and $\bigcap_{A\in\mathcal{A}}(B\cup A) \subseteq B\cup\left(\bigcap_{A\in\mathcal{A}} A\right)$.

Let $x\in B\cup\left(\bigcap_{A\in\mathcal{A}} A\right)$. Then ($x\in B$) or ($x\in A$ for all $A\in\mathcal{A}$). In either case,

we have $x\in(B\cup A)$ for all $A\in\mathcal{A}$, so $x\in\bigcap_{A\in\mathcal{A}}(B\cup A)$.

Let $x\in\bigcap_{A\in\mathcal{A}}(B\cup A)$. Then $x\in B\cup A$ for all $A\in\mathcal{A}$.

Either $x\in A$ for all $A\in\mathcal{A}$, or there exist an $A'\in\mathcal{A}$ with $x\notin A'$.

But $x\in B\cup A'$, so then $x\in B$. We have $x\in B$ or $x\in\bigcap_{A\in\mathcal{A}} A$ i.e. $x\in B\cup\left(\bigcap_{A\in\mathcal{A}} A\right)$ $\square$

⑥ Will prove $\left(\bigcup_{A\in\mathcal{A}} A\right)^c \subseteq \bigcap_{A\in\mathcal{A}} A^c$ and $\bigcap_{A\in\mathcal{A}} A^c \subseteq \left(\bigcup_{A\in\mathcal{A}} A\right)^c$.

If $x \in \left(\bigcup_{A\in\mathcal{A}} A\right)^c$, we have $x \notin \bigcup_{A\in\mathcal{A}} A$. So x cannot be in <u>any</u> $A\in\mathcal{A}$, i.e. $x \notin A$ for all $A\in\mathcal{A}$. This is the same as $x\in A^c$ for all $A\in\mathcal{A}$, i.e. $x\in \bigcap_{A\in\mathcal{A}} A^c$.

If $x\in\bigcap_{A\in\mathcal{A}} A^c$, then $x\in A^c$ for all $A\in\mathcal{A}$. So x cannot be a member of <u>any</u> $A\in\mathcal{A}$, i.e. $x\notin\bigcup_{A\in\mathcal{A}} A$. This is the same as $x\in\left(\bigcup_{A\in\mathcal{A}} A\right)^c$.

$\blacksquare$

Will prove $\left(\bigcap_{A\in\mathcal{A}} A\right)^c \subseteq \bigcup_{A\in\mathcal{A}} A^c$ and $\bigcup_{A\in\mathcal{A}} A^c \subseteq \left(\bigcap_{A\in\mathcal{A}} A\right)^c$.

If $x\in\left(\bigcap_{A\in\mathcal{A}} A\right)^c$, we have $x\notin\bigcap_{A\in\mathcal{A}} A$. So there must exist an $A'\in\mathcal{A}$ with $x\notin A'$, i.e. $x\in(A')^c$. As there exists an $A'\in\mathcal{A}$ with $x\in A'^c$, we have $x\in\bigcup_{A\in\mathcal{A}} A^c$.

If $x\in\bigcup_{A\in\mathcal{A}} A^c$, we have an $A'\in\mathcal{A}$ with $x\notin A'$. So x is not in all $A\in\mathcal{A}$, i.e. $x\notin\bigcap_{A\in\mathcal{A}} A$. This is the same as $x\in\left(\bigcap_{A\in\mathcal{A}} A\right)^c$.

$\blacksquare$

1.41

(1)

$A([-1,2]) = \{y \mid y = A(x) \text{ for an } x \in [-1,2]\}$.

Note that if $y<0$, there cannot exist an $x \in \mathbb{R}$ s.t. $x^2 = y$.
Also note that if $y > 4$, then for $x^2 = y$ to be true, we must have $x > 2$.
Lastly, for any $y \in [0,4]$, there exist an $x \in [0,2] \subseteq [-1,2]$ with $x^2 = y$. So $A([-1,2]) = [0,4]$.

$A^{-1}([-1,2]) = \{x \mid A(x) \in [-1,2]\}$.

If $|x| \leq \sqrt{2}$, we have $x^2 \in [0,2] \subseteq [-1,2]$, and if $|x| > \sqrt{2}$, we have $x^2 \notin [-1,2]$, so we have

$$A^{-1}([-1,2]) = [-\sqrt{2}, \sqrt{2}].$$

(3)

Not injective: Note that $A(-2) = A(2) = 4$, but $-2 \neq 2$.
Not surjective: Note that there exists no $x \in \mathbb{R}$ with

$$A(x) = -1.$$

The function $A(x) = x^3$, however is both injective and surjective, as we know that any real number has a <u>unique</u> third root.

(4)

Let $x, y \in \mathbb{R}$, $x \neq y$. Then either $x<y$ or $y<x$.
For simplicity, assume $x<y$. Then $A(x) < A(y)$ (strictly increasing), so $A(x) \neq A(y)$. As this is the case for any two numbers in $\mathbb{R}$, the function is injective.
It need <u>not</u> be surjective. As an example $\arctan(x)$ is strictly increasing, but $\arctan(\mathbb{R}) = (-\frac{\pi}{2}, \frac{\pi}{2})$.

⑦

a) Choose $x_1, x_2 \in X$, with $x_1 \neq x_2$. Must show $(g \circ f)(x_1) \neq (g \circ f)(x_2)$.

As f is injective, we have $f(x_1) = y_1$, $f(x_2) = y_2$ with $y_1 \neq y_2$.

As g is injective, we have $g(y_1) = z_1$, $g(y_2) = z_2$ with $z_1 \neq z_2$.
Here we use $y_1 \neq y_2$.

We now see $(g \circ f)(x_1) = g(f(x_1)) = g(y_1) = z_1 \neq z_2 = g(y_2) = g(f(x_2))$.

b) Choose $z \in Z$. Must show that there exist an $x \in X$ with $(g \circ f)(x) = z$.

As g is surjective, there exists an $y \in Y$ with $g(y) = z$.

As f is surjective, there exists an $x \in X$ with $f(x) = y$.

Then $(g \circ f)(x) = g(f(x)) = g(y) = z$.

c) As both f and g are injective, by a) $g \circ f$ must be injective.
As both f and g are surjective, by b) $g \circ f$ must be surjective.
So $g \circ f$ is bijective, so there exists an unique inverse.
Must show that $(f^{-1}) \circ (g^{-1})$ is this inverse, by showing that

$$(g \circ f) \circ ((f^{-1}) \circ (g^{-1}))(z) = z.$$

We have
$$g \circ f \circ (f^{-1}) \circ (g^{-1})(z) = g(f(f^{-1}(g^{-1}(z))))$$
$$= g(g^{-1}(z)) \quad \text{as } f(f^{-1}(y)) = y$$
$$= z \quad \text{as } g(g^{-1}(z)) = z.$$

which is exactly what we needed.

1.5

③ ~~Will~~ first rewrite the equivalence relation in an "easier to check"-way.

A line in the plane can be written either as $y=ax+b$ or as $x=a$.

Two lines are parallell if they can be written as $y=a_1x+b_1$ and $y=a_2x+b_2$, with $a_1=a_2$, or if both can be written as $x=a_1$, $x=a_2$.

Must check reflexivity, symmetry and transitivity.

Reflexivity: $\boxed{l\sim l}$. "Or equal" is built into the equivalence, so this must be true.

Symmetry: $\boxed{l\sim m \Rightarrow m\sim l}$.

Case 1: $l: y=a_1x+b_1$. As $l\sim m$, we have $m: y=a_2x+b_2$, with $a_1=a_2$. But $a_1=a_2 \Rightarrow a_2=a_1$, so changing the order doesn't matter. So $m\sim l$.

Case 2: $l: x=a_1$. As $l\sim m$, we have $m: x=a_2$. Changing the order here still does not matter, so, we have $m\sim l$.

Transitivity: $\boxed{l\sim m \text{ and } m\sim n \Rightarrow l\sim n}$.

Case 1: $l: y=a_1x+b_1$. As $l\sim m$, we have $m: y=a_2x+b_2$, with $a_1=a_2$.

As $m\sim n$, we have $n: y=a_3x+b_3$, with $a_2=a_3$.

So $a_1=a_2$ and $a_2=a_3$, which means $a_1=a_3$.

Therefore, we have $l\sim n$.

Case 2: $l: x=a_1$. As $l\sim m$, we have $m: x=a_2$.

As $m\sim n$, we have $n: x=a_3$.

As l and n are both on the form $x=a$, we have $l\sim n$.

5

All these came from reflexivity, symmetry and transitivity of $=$.

Reflexive: $\boxed{(x,y,z)\sim(x,y,z)}$

$$3x-y+2z = 3x-y+2z \qquad \text{True.}$$

Symmetric: $\boxed{(x,y,z)\sim(x',y',z') \Rightarrow (x',y',z')}$

If $3x-y+2z = 3x'-y'+2z'$, then

$3x'-y'+2z' = 3x-y+2z$. So True.

Transitive: $\boxed{(x,y,z)\sim(x',y',z') \wedge (x',y',z')\sim(x'',y'',z'') \Rightarrow (x,y,z)\sim(x'',y'',z'')}$

If $3x-y+2z = 3x'-y'+2z'$ and

$3x'-y'+2z' = 3x''-y''+2z''$, then we have

$3x-y+2z = 3x''-y''+2z''$. So True.

What do the equivalence classes look like?

If $[(x',y',z')]$ is an equivalence class, let $3x'-y'+2z'=k$.

Then, if $(x,y,z)\in[(x',y',z')]$ we have $(x,y,z)\sim(x',y',z')$,

i.e. $3x-y+2z=k$ as well.

So we have $[(x',y',z')] = \{(x,y,z)\in\mathbb{R}^3 \mid 3x-y+2z=k\}$,

which we recognize as the set of points in the plane with normal vector $(3,-1,2)$ through (x',y',z').

So each equivalence class is a plane with normal vector $(3,-1,2)$ and $\mathbb{R}^3/\sim \; = \{[(x,y,z)] \mid (x,y,z)\in\mathbb{R}^3\}$ is the set of parallel planes, all normal to $(3,-1,2)$.

6

Reflexive: $\boxed{x \sim x}$

$x - x = 0$, and 0 is divisible by m $(0 = 0 \cdot m)$.
So True.

Symmetric: $\boxed{x \sim y \Rightarrow y \sim x}$

If $x \sim y$, then $x - y = k \cdot m$, for $k \in \mathbb{Z}$, a whole number.
Then $y - x = -(x-y) = -k \cdot m = (-k) \cdot m$, so $y - x$ is also divisible by m. Therefore, $y \sim x$, as we needed.

Transitive: $\boxed{x \sim y \wedge y \sim z \overset{?}{\Rightarrow} x \sim z}$

If $x \sim y$, then $x - y = k \cdot m$, $k \in \mathbb{Z}$.
If $y \sim z$, then $y - z = l \cdot m$, $l \in \mathbb{Z}$.
Now $x - z = x - y + y - z = (x-y) + (y-z) = k \cdot m + l \cdot m = (k+l) \cdot m$.
So $x - z$ is divisible by m, and we have $x \sim z$, as needed.

Any $x \in \mathbb{Z}$ can be written as $k \cdot m + r$, where $0 \le r < m$.
And if $x = k \cdot m + r$, then $x - r = k \cdot m$, i.e. $x \sim r$.

So for any $x \in \mathbb{Z}$, we have $x \in [r]$ for $0 \le r < m$.
The number of equivalence classes are therefore m, $(0, 1, \dots, m-1)$ and they will be on this form:

$[0] = \{\dots, -3m, -2m, -m, 0, m, 2m, 3m, \dots\}$

$[1] = \{\dots, -3m+1, -2m+1, -m+1, 1, m+1, 2m+1, 3m+1, \dots\}$

$[2] = \{\dots, -3m+2, -2m+2, -m+2, 2, m+2, 2m+2, 3m+2, \dots\}$

$\vdots$

$[m-1] = \{\dots, -3m+m-1, -2m+m-1, -m+m-1, m-1, m+m-1, 2m+m-1, 3m+m-1, \dots\}$

$= \{\dots, -2m-1, -m-1, -1, m-1, 2m-1, 3m-1, 4m-1, \dots\} = [-1]$.