1.6

(1)

Let $A \subseteq B$, B countable. Then we can write

$$B = \{b_1, b_2, \dots, b_n, \dots\}$$

Let n_1 be the first natural number s.t. $b_{n_1} \in A$,
n_2 be the first natural number with $n_2 > n_1$ and $b_{n_2} \in A$,
n_3 s.t. $n_3 > n_2$ $b_{n_3} \in A$, etc.

Now let $a_i = b_{n_i}$. If there are only finitely many such a_i, just repeat the last one ad infinitum.

We now have $\{a_1, \dots, a_k, \dots\} \subseteq A$, as each element $a_i \in A$ by definition, and $A \subseteq \{a_1, \dots, a_k, \dots\}$, as each element in A must be equal to a $b_n = b_{n_j}$ for a $j \in \mathbb{N}$ (The construction will not "skip" an element in A).

So we have

$$A = \{a_1, a_2, \dots, a_k, \dots\},$$

and A is therefore countable.

(2)

As A_i is countable, we can write

$$A_i = \{a_1^i, a_2^i, a_3^i, \dots\}.$$

Then, for each $k \geq n$, we can list all the elements

$$(a_{k_1}^1, a_{k_2}^2, \dots, a_{k_n}^n) \text{ with } k_1 + k_2 + \dots + k_n = k.$$

There is a finite amount of elements for a given k.

We can then concatenate these lists for increasing k's, to get

$$A_1 \times A_2 \times \dots \times A_n = \{(a_1^1, a_1^2, \dots, a_1^n), (a_2^1, a_1^2, \dots, a_2^n), (a_1^1, \dots, a_2^{n-1}, a_1^n), \dots\},$$

so $A_1 \times A_2 \times \dots \times A_n$ is countable. $\blacksquare$

Alternative proof:

Induction: If A_1 and A_2 are countable, then $A_1 \times A_2$ is countable by Prop 1.6.1.

So true for $n = 1, 2$.

Assume it's true for $n \leq k$.

Then, for $n = k+1$, we have

$$A_1 \times A_2 \times \dots \times A_k \times A_{k+1} = (A_1 \times A_2 \times \dots A_k) \times A_{k+1}.$$

By induction hypothesis, we have that $A_1 \times \dots \times A_k$ is countable, and as both $A_1 \times A_2 \times \dots \times A_k$ and A_{k+1} are countable, we have that $A_1 \times A_2 \times \dots \times A_k \times A_{k+1}$ is countable by Prop 1.6.1. $\blacksquare$

(4) Show: $\mathcal{P}(\mathbb{N}) = \{A \mid A \subseteq \mathbb{N}\}$ is uncountable.

Assume that $\mathcal{P}(\mathbb{N})$ is countable,

$$\mathcal{P}(\mathbb{N}) = \{A_1, A_2, \dots, A_n, \dots\}.$$

Will construct a $B \in \mathcal{P}(\mathbb{N})$ with $B \neq A_i$ for all i.

We construct B the following way:

For each $n \in \mathbb{N}$, we let $n \in B$ if $n \notin A_n$, and $n \notin B$ otherwise.

So $1 \in B$ if $1 \notin A_1$, $1 \notin B$ if $1 \in A_1$, etc.

Then we have $B \neq A_i$ for any i, as either $i \notin A_i$, and then $i \in B$, or $i \in A_i$, and then $i \notin B$.

So we have $B \in \mathcal{P}(\mathbb{N})$, but $B \neq A_i$ for all i, i.e. B is <u>not</u> in the list. This is a contradiction, as we assumed $\mathcal{P}(\mathbb{N}) = \{A_1, A_2, A_3, \dots, A_n, \dots\}$

3.11

① Example 4:

i) Positivity:

a) $d(x,y) \geq 0$:

As $d(x,y) =$ # indices n s.t. $x_n \neq y_n$, we cannot get a negative number.

b) $d(x,y)=0$ iff $x=y$:

If $x=y$, then there are no indices where $x_n \neq y_n$, so $d(x,y)=0$, and if $d(x,y)=0$, i.e. $x_n=y_n$ for all n, then $x=y$.

ii) Symmetry:

Whenever $x_n \neq y_n$, we have $y_n \neq x_n$, so

$$
\begin{aligned}
d(x,y) &= \#\text{indices where } x_n \neq y_n \\
&= \#\text{ indices where } y_n \neq x_n \\
&= d(y,x)
\end{aligned}
$$

iii) Triangle inequality:

If $x_n \neq y_n$, we can't have both $x_n = z_n$ and $z_n = y_n$, so the set of indices where $x_n \neq y_n$ must be contained in the union of the sets where $x_n \neq z_n$ and $z_n \neq y_n$,

so
$$
\begin{aligned}
d(x,y) &= \#\{n \mid x_n \neq y_n\} \leq \#(\{n \mid x_n \neq z_n\} \cup \{n \mid z_n \neq y_n\}) \\
&\leq \#\{n \mid x_n \neq z_n\} + \#\{n \mid z_n \neq y_n\} \\
&= d(x,z) + d(z,y).
\end{aligned}
$$

■

② Example 5, d_1.

<u>Note:</u> As $(f-g):[a,b]\to\mathbb{R}$ is continuous, we know from Thm 2.3.4. we know that
$\sup\{|f(x)-g(x)|\ |x\in[a,b]\}\neq\infty$.
so $d(f,g)\in\mathbb{R}$.

i) <u>Positivity:</u>

a) $d(x,y)\geq 0$

As $d(x,y)=\sup\{|f(x)-g(x)|: x\in[a,b]\}$,
we are taking the supremum over a set of positive values, so we get something positive.

b) $d(f,g)=0$ iff $f=g$.

If $f=g$, then $f(x)=g(x)$ for all $x\in[a,b]$,
so $|f(x)-g(x)|=0$ for all $x\in[a,b]$, and the supremum is then 0.
If the supremum is 0, we have $|f(x)-g(x)|\leq 0$ for all $x\in[a,b]$, i.e. $f(x)-g(x)=0$ for all $x\in[a,b]$,
so $f(x)=g(x)$ for all $x\in[a,b]$ $\Rightarrow$ $f=g$.

ii) <u>Symmetry:</u>

We have $|f(x)-g(x)|=|-1||g(x)-f(x)|=|g(x)-f(x)|$,
so $d(f,g)=\sup\{|f(x)-g(x)|: x\in[a,b]\}=\sup\{|g(x)-f(x)|: x\in[a,b]\}$
$=d(g,f)$

iii) <u>Triangle inequality:</u>

We have $|f(x)-g(x)|=|f(x)-h(x)+h(x)-g(x)|\leq|f(x)-h(x)|+|h(x)-g(x)|$
So $d(f,g)=\sup\{|f(x)-g(x)|: x\in[a,b]\}\leq\sup\{|f(x)-h(x)|+|h(x)-g(x)|: x\in[a,b]\}$
$\leq\sup\{|f(x)-h(x)|: x\in[a,b]\}+\sup\{|h(x)-g(x)|\}$
$=d(f,h)+d(h,g)$

③ Example 5, d_2

Note: Continuous functions on a finite interval have finite integrals, so $d(f,g) \in \mathbb{R}$.

Positivity:

ⓐ $d(f,g) \geq 0$

The integral of a non-negative function is non-negative, so $\int_a^b |f(x)-g(x)|dx = d(f,g) \geq 0$.

ⓑ $d(f,g)=0$ iff $f=g$.

If $f=g$, then $f(x)-g(x)=0$, so $d(f,g) = \int_a^b 0\, dx = 0$.

If $d(f,g)=0$, then $\int_a^b |f(x)-g(x)|dx = 0$, and the only way a non-negative function can have zero integral is if it's constantly zero, i.e. $f(x)-g(x)=0$ for all $x \in [a,b]$. So $f=g$.

Symmetry:

$|f(x)-g(x)| = |-1||g(x)-f(x)| = |g(x)-f(x)|$, so

$d(f,g) = \int_a^b |f(x)-g(x)|dx = \int_a^b |g(x)-f(x)|dx = d(g,f)$.

Triangle Inequality:

We have $|f(x)-g(x)| = |f(x)-h(x)+h(x)-g(x)| \leq |f(x)-h(x)| + |h(x)-g(x)|$,

so $d(f,g) = \int_a^b |f(x)-g(x)|dx \leq \int_a^b |f(x)-h(x)| + |h(x)-g(x)|dx$

$= \int_a^b |f(x)-h(x)|dx + \int_a^b |h(x)-g(x)|dx$

$= d(f,h) + d(h,g)$

⑥

Positivity:

a) $d(x,y) \geq 0$

As $d(x,y) = \|x-y\|$ and $\|x\| \geq 0$ for all x, we have $\|x-y\| \geq 0$ as well.

b) $d(x,y) = 0$ iff $x=y$.

If $x=y$, then $x-y=0$, so $\|x-y\| = \|0\| = 0$, and if $\|x-y\|=0$, then we must have $x-y=0$, i.e. $x=y$.

Symmetry:

We have $d(x,y) = \|x-y\| = \|(-1)(y-x)\| = |-1|\|y-x\|$
$= \|y-x\| = d(y,x)$.

Triangle Inequality:

We have $d(x,y) = \|x-y\| = \|x-z+z-y\| \leq \|x-z\| + \|z-y\| = d(x,z) + d(z,y)$.

⑦ Induction.

We know the statement is true for $n=2$ by the triangle inequality.

Assume it's true for $n=k$.

Show that it's true for $n=k+1$.

So we have $d(x_1, x_k) \leq d(x_1,x_2) + \dots + d(x_{k-1}, x_k)$ by induction.

Therefore, we have

$$d(x_1, x_{k+1}) \leq d(x_1,x_k) + d(x_k, x_{k+1}) \quad \text{Triangle ineq.}$$
$$\leq d(x_1,x_2) + \dots + d(x_{k-1},x_k) + d(x_k, x_{k+1}) \quad \text{Ind. Hyp.}$$

which is what we wanted.

▨

⑧

a) $x_n \to x$ as $n \to \infty$ implies that for any $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ s.t. $d(x_n, x) < \varepsilon$ whenever $n \geq N$.

We want to show that for any $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ s.t. $|d(x_n, y) - d(x, y)| < \varepsilon$ whenever $n \geq N$.

By the inverse triangle inequality we have

$$|d(x_n, y) - d(x, y)| \leq d(x_n, x) < \varepsilon$$ when $n \geq N$, as needed

$\blacksquare$

b) $x_n \to x$ as $n \to \infty$ implies that for any $\frac{\varepsilon}{2} > 0$ there exists an $N_1 \in \mathbb{N}$ s.t. $d(x_n, x) < \frac{\varepsilon}{2}$ when $n \geq N_1$, and

$y_n \to y$ as $n \to \infty$ implies that for any $\frac{\varepsilon}{2} > 0$, there exists an $N_2 \in \mathbb{N}$ s.t. $d(y_n, y) < \frac{\varepsilon}{2}$ when $n \geq N_2$.

We then have

$$
\begin{aligned}
|d(x_n, y_n) - d(x, y)| &= |d(x_n, y_n) - d(x_n, y) + d(x_n, y) - d(x, y)| \\
&\leq |d(x_n, y_n) - d(x_n, y)| + |d(x_n, y) - d(x, y)| \\
&\leq d(y_n, y) + d(x_n, x) \qquad \text{Inverse Triangle Ineq.} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}
$$

whenever $n \geq N = \max(N_1, N_2)$.

$\blacksquare$

3.2

①

If there is an N s.t. $x_n = a$ for all $n \geq N$, we have

$d(x_n, a) = d(a,a) = 0 < \varepsilon$ for all $n \geq N$, no matter what ε is ($\varepsilon > 0$),

so we have $x_n \to a$.

If $x_n \to a$ we have that for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ s.t $d(x_n, a) < \varepsilon$ whenever $n \geq N$.

Choose $\varepsilon = 1/2$, and let N be the corresponding natural number.

Then $d(x_n, a) < \frac{1}{2}$ for all $n \geq N$. But in the discrete metric, we have $d(x,a) = 1$ if $x \neq a$, $d(x,a) = 0$ if $x = a$. So $d(x,a) < 1/2$ implies $d(x,a) = 0$, i.e. $x = a$. So we have $x_n = a$ for all $n \geq N$.

②

Choose an $\varepsilon > 0$. As g is continuous at $f(a)$, we can find a δ_1 s.t.

$$d_Z(g(y), g(f(a))) < \varepsilon \text{ whenever } d_Y(y, f(a)) < \delta_1.$$

Given this δ_1, as f is continuous at a, we can find a δ s.t.

$$d_Y(f(x), f(a)) < \delta_1 \text{ whenever } d_X(x,a) < \delta.$$

Then, if $d_X(x,a) < \delta$, we have $d_Y(f(x), f(a)) < \delta_1$,

so $d_Z(g(f(x)), g(f(a))) < \varepsilon$.

So we have $d_Z(h(x), h(a)) < \varepsilon$ whenever $d_X(x,a) < \delta$,

so h is continuous at a.

③ We have the metric spaces (X, d_X) and (Y, d_Y), with $A \subseteq X$. Let us look at the metric space (A, d_A), where $d_A = d_X|_{A \times A}$, d_X restricted to input-values from A.

<u>Def</u> (3.2.8) A function $f: X \to Y$ is continuous at a point $a \in A \subseteq X$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ s.t. $d_Y(f(x), f(a)) < \varepsilon$ whenever $x \in A$ and $d_X(x, a) < \delta$.

<u>Def</u> (3.2.4) A function $f: A \to Y$ is continuous at a point $a \in A$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ s.t. $d_Y(f(x), f(a)) < \varepsilon$ whenever $d_A(x, a) < \delta$.

Note that $x \in A$ is implicit in the second definition, as A is our source. And if $x \in A$ we have $d_A(x,a) = d_X(x,a)$. So these definitions say the same thing.

Therefore;

3.2.9 i) f continuous at $a \in A$ (with d_X)

$\Updownarrow$

3.2.5 i) $f: A \to Y$ continuous at $a \in A$ (with d_A)

$\Updownarrow$ By 3.2.5

3.2.5 ii) For all sequences $\{x_n\}$ in A converging to a, the sequence $\{f(x_n)\}$ converges to $f(a)$ (with d_A)

$\Updownarrow$

3.2.9 ii) For all sequences $\{x_n\}$ in A converging to a, the sequence $\{f(x_n)\}$ converges to $f(a)$

As $d_A(x_n, a) = d_X(x_n, a)$ when $x_n \in A, a \in A$, the sequence will converge in (X, d_X) iff it converges in (A, d_A).

(5)

Want to show $d(x,a)$ continuous, i.e. given $a, b \in X$, we have $f_a(x) = d(x,a)$ continuous at b.

Choose any $\varepsilon > 0$. Must find a $\delta > 0$ s.t.

$$|f_a(x) - f_a(b)| < \varepsilon \quad \text{whenever } d(x,b) < \delta.$$

We have $|f_a(x) - f_a(b)| = |d(x,a) - d(b,a)| \leq d(x,b)$ by the Inverse Triangle Ineq.

So by choosing $\delta = \varepsilon$, we have

$$|f_a(x) - f_a(b)| \leq d(x,b) < \delta = \varepsilon.$$

$\blacksquare$

(6) Want to show $f(x)$ continuous, i.e. given $a \in X$, we have $f(x)$ continuous at a.

Choose any $\varepsilon > 0$. Must find a $\delta > 0$ s.t.

$$d_Y(f(x), f(a)) < \varepsilon \text{ whenever } d_X(x,a) < \delta.$$

By Lipschitz, we have $d_Y(f(x), f(a)) \leq K \cdot d_X(x,a) < K \cdot \delta$.

So by choosing $\delta = \frac{\varepsilon}{K}$, we get $d_Y(f(x), f(a)) < \varepsilon$ whenever $d_X(x,a) < \delta$, and f is continuous at a. This is ~~true~~ for all a.

<u>Note</u>! This only works if $K > 0$.

Luckily, $K < 0$ cannot happen, as we'd have $d_Y(f(x), f(a)) < 0$, contradicting positivity, and if $K = 0$, we have $d(f(x), f(a)) = 0 < \varepsilon$ for any ε, so f is still continuous.

3.3

①

a) $B(a;r) = \{x \mid d(x,a) < r\}$.

In the discrete metric space, we have $d(x,a) = 1$ if $x \neq a$ and $d(x,a) = 0$ if $x = a$.

So if $r \leq 1$, $B(a;r) = \{x \mid d(x,a) < r \leq 1\} = \{x \mid d(x,a) = 0\}$
$= \{a\}$.

And if $r > 1$, $B(a;r) = \{x \mid d(x,a) < r\} = \{x \mid d(x,a) \leq 1\}$
$= X$.

b) A boundary point a for a set A is a point where, no matter which ε you choose, $B(a;\varepsilon)$ will contain points in A and points outside of A. But as we have seen that $B(a;\varepsilon)$ is a singleton for $\varepsilon \leq 1$, $B(a,\varepsilon) = \{a\}$, so we cannot have any boundary points for any sets A.

And as an open set is a set which contains none of its boundary points, (of which we have 0), A must be open. And as a closed set is a set which contains all of its boundary points (all 0 of them), A must be closed. This is true for any $A \subseteq X$, so all sets are open and closed (clopen).

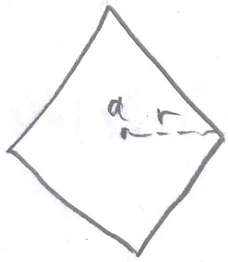
c) Given an $\varepsilon > 0$, choose $\delta = \frac{1}{2}$. Then $d(x,a) < \delta \Rightarrow x = a$, so $d_Y(f(x), f(a)) = d_Y(f(a), f(a)) = 0 < \varepsilon$ whenever $d(x,a) < \delta$, and we have that f is continuous at a. As this is true for all $a \in X$, we have that f is continuous.

②

If $d(x,y)=r$, the sum of the movement along the x-axis and the y-axis should be r, as if you had to move on a grid.

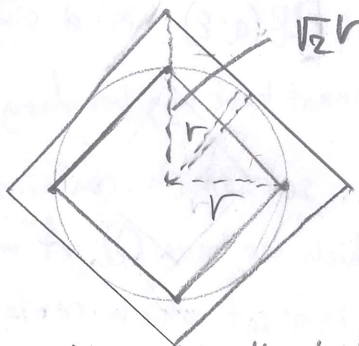
You can find a map of Manhattan to see <u>why</u> it's called the Manhattan metric.

A typical Manhattan ball looks like this:



The trick to show that the open sets are the same is to note that $|x_2-x_1|+|y_2-y_1| \leq \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \leq \sqrt{2}\left(|x_2-x_1|+|y_2-y_1|\right)$

"Proof by drawing":



So if $B_M(a,r)$ is the Manhattan ball, we have

$$B_M(a,r) \subseteq B(a,r) \subseteq B_M(a, \tfrac{\sqrt{2}}{2} r).$$

Then, if a is a boundary point of a set A in the Manhattan-metric, we have that for a given ε, $B_M(a,\varepsilon)$ has points both in and outside A. As $B(a,\varepsilon) \supseteq B_M(a,\varepsilon)$, $B(a,\varepsilon)$ must have points both in and outside A as well. As this is true for all ε, a is a boundary point for A in the <u>usual metric as well.</u>

And if a is a boundary point of A in the usual metric, we have that for a given ε, $B(a,\frac{1}{\sqrt{2}}\varepsilon)$ contains points both in and outside A. Then, as $B_M(a,\varepsilon) \supseteq B(a,\frac{1}{\sqrt{2}}\varepsilon)$, $B_M(a,\varepsilon)$ must have points both in and outside A. As this is true for all ε a must be a boundary point for A in <u>the Manhattan-metric as well.</u>

Now, as the boundary points are the same in both metrics, the open sets must be the same as well.

③ Will show that $\sup F$ and $\inf F$ must be boundary points of F, and therefore contained in F as F is closed.

Sup F:

Let $s = \sup F$, i.e. $s \geq x$ for all $x \in F$
and $s \leq d$ for all d s.t. $d \geq x$ for all $x \in F$.

Choose a small ball around s, $B(s, \varepsilon) = (s-\varepsilon, s+\varepsilon)$.
Must show that there exists points both in F and outside of F in $(s-\varepsilon, s+\varepsilon)$.

Outside of F:

Let $a = s + \frac{\varepsilon}{2}$. As $s \geq x$ for all $x \in F$ and $a > s$, we have $a \notin F$, and we have $a \in (s-\varepsilon, s+\varepsilon)$.

In F:

Let $b = s - \frac{\varepsilon}{2}$. Will show that there must exist an $a \in [s-\frac{\varepsilon}{2}, s]$ with $a \in F$. For if there did not exist an $a \in [s-\frac{\varepsilon}{2}, s]$ with $a \in F$, we would have $b \geq x$ for all $x \in F$ with $b < s$. This contradicts $s = \sup F$. So we have $a \in [s-\frac{\varepsilon}{2}, s] \subseteq (s-\varepsilon, s+\varepsilon)$ with $a \in F$.

Therefore $\sup F$ is a boundary point, and as F contains all of its boundary points, we have $\sup F \in F$.

Inf F: Same procedure, just flip the signs.

④ Will prove that $\overline{B}(a;r)^c$ is open instead. (Same thing, by P. 3.3.4)

Choose a point $x \notin \overline{B}(a;r)$ i.e. $d(x,a) > r$.

Will show that there exists a ball $B(x,\varepsilon)$ s.t. $B(x,\varepsilon) \subseteq \overline{B}(a,r)^c$.

Choose $\varepsilon = \dfrac{d(x,d)-r}{2}$, and choose a $z \in B(x,\varepsilon)$.

Then we have

$$\begin{aligned}
d(a,z) &\geq |d(a,x)-d(x,z)| && \text{by Inverse Triangle Inequality.} \\
&\geq d(a,x)-d(x,z) && \text{Removing the } |\cdot| \text{ will not give a larger value} \\
&> d(a,x)-\varepsilon && d(x,z)<\varepsilon, \text{ so } -d(x,z)>-\varepsilon. \\
&= d(a,x)-\frac{d(a,x)-r}{2} \\
&= \frac{d(a,x)+r}{2} && d(a,x)>r. \\
&> \frac{r+r}{2} = r
\end{aligned}$$

So $z \notin \overline{B}(a,r)$. This is true for any $z \in B(x,\varepsilon)$, so we have $B(x,\varepsilon) \subseteq \overline{B}(a,r)^c$. This is true for any $x \in \overline{B}(a,r)^c$, so $\overline{B}(a,r)^c$ is open $\Rightarrow$ $\overline{B}(a,r)$ is closed.

⑤ Let U be an open set in Z.

Then $(g\circ f)^{-1}(U) = \{x \in X : (g\circ f)(x) \in U\} = \{x \in X : g(f(x)) \in U\}$
$= \{x \in X : f(x) \in g^{-1}(U)\} = \{x \in X : x \in f^{-1}(g^{-1}(U))\}$

So $(g\circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$. And $g^{-1}(U)$ is open, as g is continuous. Therefore, $f^{-1}(g^{-1}(U))$ is open, as f is continuous.

Which shows that if U is open in Z, $(g\circ f)^{-1}(U)$ is open as well

$\Rightarrow$ $g\circ f$ is continuous.