$\boxed{3.3}$

7. A° is all interior points of A.

Must show that all points in A° are actually interior points of A°, not just interior points of A.

Let $x \in A^\circ$. Then is an interior point of A, so there exists an $\varepsilon > 0$ s.t. $B(x,\varepsilon) \subseteq A$. Will show $B(x,\varepsilon) \subseteq A^\circ$.

Choose any $z \in B(x,\varepsilon)$. As $z \in A$, we have that either $z \in A^\circ$ or z is a boundary point of A. Let us show that z cannot be a boundary point.

If z is a boundary point of A, then $B(z,\delta)$ will contain both points in A and outside of A, for all $\delta > 0$.

Choose $\delta = \dfrac{\varepsilon - d(x,z)}{2}$. Then any point $w \in B(z,\delta)$

has
$$d(x,w) \le d(x,z) + d(z,w) < d(x,z) + \frac{\varepsilon - d(x,z)}{2}$$
$$= \frac{\varepsilon}{2} + \frac{d(x,z)}{2} < \varepsilon,$$
so $w \in B(x,\varepsilon) \subseteq A$,

and we cannot have any points in $B(z,\delta)$ outside of A. So z is not a boundary point of A, and we have that x is an interior point of A°. A° is therefore open. ∎

8

a) Must show that $\overline{A}$ contains all of its boundary points, i.e. that a boundary point for $\overline{A}$ is a boundary point for A.

Choose a boundary point for $\overline{A}$, x. Then $B(x,\varepsilon)$ contains points in $\overline{A}$ and outside of $\overline{A}$, for all ε.

As $A \subseteq \overline{A}$, any point outside of $\overline{A}$ is also outside of A, so $B(x,\varepsilon)$ contains points outside of A.

Choose any point $z \in B(x,\varepsilon)$ with $z \in \overline{A}$. Then we either have $z \in A$, or z a boundary point of A.

If $z \in A$, $B(x,\varepsilon)$ has points both in A and outside of A, so x is a boundary point of A, and $x \in \overline{A}$.

If z is a boundary point of A, we have that $B(z,\delta)$ contains points in A and outside of A for all $\delta > 0$.

Choose $\delta = \frac{\varepsilon - d(x,z)}{2}$. As $B(z,\delta)$ contains a point in A, choose such a point, call it w. Then we have

$$d(x,w) \leq d(x,z) + d(z,w) < d(x,z) + \frac{\varepsilon - d(x,z)}{2}$$
$$= \frac{d(x,z)}{2} + \frac{\varepsilon}{2} < \frac{\varepsilon + \varepsilon}{2} = \varepsilon.$$

So $w \in B(x,\varepsilon)$, $w \in A$. We have that $B(x,\varepsilon)$ contains points both in and outside of A, so x is a boundary point of A, i.e. $x \in \overline{A}$. So $\overline{A}$ contains all of its boundary points and is therefore closed. $\boxed{}$

b) As $\{a_n\} \subset A \subseteq \overline{A}$, we have that $\{a_n\}$ is a sequence in $\overline{A}$. As $\overline{A}$ is closed, $a \in \overline{A}$ by Prop 3.3.6. $\boxed{}$

1)

a)

We have (for $x \in A$) $B_A(x;r) = \{y \in A : d_A(x,y) < r\}$

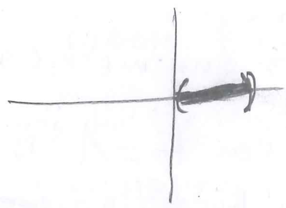
$= \{y \in A : d(x,y) < r\}$ as $d_A(x,y) = d(x,y)$ for $x,y \in A$.

$= B(x,r) \cap A.$

So if $G \subseteq A$ open in (X,d), we have that for all $x \in G$, there exists an $\varepsilon > 0$ s.t. $B(x,\varepsilon) \subseteq G$.
As $B_A(x,r) = B(x,r) \cap A \subset B(x,r)$, we have that $B_A(x,\varepsilon) \subseteq G$ as well, so x is an interior point wrt the d_A-metric as well, and G is open in (A,d_A).

b)

Let $X = \mathbb{R}^2$, $A = \mathbb{R} \times \{0\}$, the x-axis.
Then $(0,1) \times \{0\}$ is open in (A,d_A), but closed in X.



c)

Must show $G \subseteq A$ open in (A,d_A), A open in (X,d) $\Rightarrow$ G open in (X,d). Other way was shown in a).
Choose an $x \in G$. As G is open in (A,d_A), there exists an ε_1 s.t. $B_A(x;\varepsilon_1) \subseteq G$. As A is open in (X,d), there exists an ε_2 s.t. $B(x,\varepsilon_2) \subseteq A$. Let $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$.
Then we have $B_A(x,\varepsilon_1) \cap B(x,\varepsilon_2) \subset G$, with

$$B_A(x,\varepsilon_1) \cap B(x,\varepsilon_2) = B(x,\varepsilon_1) \cap A \cap B(x,\varepsilon_2) = B(x,\varepsilon_1) \cap B(x,\varepsilon_2)$$
$$= B(x,\varepsilon),$$

so $B(x,\varepsilon) \subseteq G$. This can be done for any $x \in G$, so G is open in (X,d).

(10)

a) Will show that if x is a boundary point of F in (A, d_A), then x is a boundary point in (X,d) as well. As F is closed in (X,d), we then have $x \in F$, so F contains all of its A-boundary points, and is closed in A.

Let $x \in A$ be a boundary point of F in (A, d_A), i.e. for any choice of ε, we have that $B_A(x,\varepsilon) = B(x,\varepsilon) \cap A$ contains both points in F and points in $A \setminus F$.

As $B_A(x,\varepsilon) = B(x,\varepsilon) \cap A \subseteq B(x,\varepsilon)$, we have that $B(x,\varepsilon)$ contains both points in F and points in $A \setminus F \subseteq X \setminus F = F^c$.

As this is true for any ε, we have that x is a boundary point of F in (X,d) as well, and this is what we needed. $\boxed{\text{M}}$

b) For any metric space, let A be a non-closed set in X. Then $F = A$ is closed in A (any set is closed in itself), but by definition of A not closed in X.

c) Have shown F closed in $X \Rightarrow F$ closed in A. Must show other way. Will prove this contrapositively, i.e. will show F not closed in $X \Rightarrow F$ not closed in A.

If F is not closed in X, there exists an $x \in X \setminus F$ with x a boundary point of F. So for any $\varepsilon > 0$, $B(x,\varepsilon)$ will contain points in F and points outside of F.

If $x \in A$, we have that $B_A(x,\varepsilon) = B(x,\varepsilon) \cap A$ contains points in F and points in $A \setminus F$ (x itself, if nothing else), so F has a boundary point in A that is not in F, so F is not closed.

But if $x \notin A$, we have that $B(x,\varepsilon)$ contains points both in A ($F \subseteq A$) and outside of A (x itself), so x must be a boundary point of A. And A is closed, and contains all of its boundary points, so $x \in A$, a contradiction. I.e. x must be in A, and F is therefore not closed, as needed. $\boxed{\text{M}}$

(12) a) Let $x \in \bigcup_{G \in \mathcal{G}} G$. Will show that x is an interior point.

As $x \in \bigcup_{G \in \mathcal{G}} G$, there exists a $G \in \mathcal{G}$ with $x \in G$.

As G is open, there exists an $\varepsilon > 0$ s.t. $B(x,\varepsilon) \subseteq G \subseteq \bigcup_{G \in \mathcal{G}} G$.

So, as we have an $\varepsilon > 0$ s.t. $B(x,\varepsilon) \subseteq \bigcup_{G \in \mathcal{G}} G$, x is an interior point, and as this works for any x, we have that $\bigcup_{G \in \mathcal{G}} G$ is open.

b) Let $x \in G_1 \cap \dots \cap G_n$. Will show that x is an interior point.

As G_i is open, there exists an $\varepsilon_i > 0$ s.t. $B(x,\varepsilon_i) \subseteq G_i$.

Let $\varepsilon = \min_i (\varepsilon_i)$. Then $B(x,\varepsilon) \subseteq B(x,\varepsilon_i) \subseteq G_i$ for all i,

so $B(x,\varepsilon) \subseteq G_1 \cap \dots \cap G_n$, which shows that x is an interior point. As this works for all x, $G_1 \cap \dots \cap G_n$ is open.

Ex: We have $\{0\} = \bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right)$ from Exercise 1.3.2.

This is an infinite intersection of open sets, but the result is not open.

(13) Will use De Morgans law (Exer. 1.3.5/6) and Prop 3.3.11 to prove this.

a) $F \in \mathcal{F}$ are all closed $\Rightarrow F^c$ is open for all $F \in \mathcal{F}$.

$\Rightarrow \bigcup_{F \in \mathcal{F}} F^c$ is open by Prop 3.3.11(a) $\Rightarrow \left(\bigcup_{F \in \mathcal{F}} F^c\right)^c$ is closed

$\Rightarrow \bigcap_{F \in \mathcal{F}} F^{cc}$ is closed by De Morgans law $\Rightarrow \bigcap_{F \in \mathcal{F}} F$ is closed by $F^{cc} = F$.

b) $F_1, \dots, F_n$ are all closed $\Rightarrow F_i^c$ is open for $i = 1, \dots, n$.

$\Rightarrow F_1^c \cap \dots \cap F_n^c$ is open by Prop 3.3.11 (b). $\Rightarrow (F_1^c \cap \dots \cap F_n^c)^c$ is closed

$\Rightarrow F_1^{cc} \cup \dots \cup F_n^{cc}$ is closed by De Morgans law $\Rightarrow F_1 \cup \dots \cup F_n$ is closed by $F_i^{cc} = F_i$.

Ex: We have $(0,1] = \bigcup_{n \in \mathbb{N}} \left[\frac{1}{n}, 1\right]$ by Exercise 1.3.3.

This is an infinite union of closed sets, but the result is not closed.

4

① Will show that all Cauchy-sequences converge.

Let $\{x_n\}$ be a Cauchy-sequence, and choose $\varepsilon = 1/2$.
Then there exists an $N \in \mathbb{N}$ s.t. $d(x_n, x_m) < \varepsilon$ whenever $n, m \geq N$,
by the criterion for Cauchy sequences.
In the discrete metric, $d(x_n, x_m) < \frac{1}{2} \Rightarrow d(x_n, x_m) = 0 \Rightarrow x_n = x_m$.
So we have that if we call $x_N = a$, $x_n = a$ for all $n \geq N$, and
the sequence converges to a. ∎

② Let $\{(x_n, y_n)\}$ be a Cauchy-sequence. Then for any $\varepsilon > 0$,
there exists an $N \in \mathbb{N}$ s.t. $d((x_n, y_n), (x_m, y_m)) = d(x_n, x_m) + d(y_n, y_m) < \varepsilon$
whenever $n, m \geq N$. As both $d(x_n, x_m) \leq d(x_n, x_m) + d(y_n, y_m)$ and
$d(y_n, y_m) \leq d(x_n, x_m) + d(y_n, y_m)$, we have that
$\{x_n\}$ must be a Cauchy-sequence in X and $\{y_n\}$ a Cauchy-sequence in Y.
As both X and Y are complete, $\{x_n\}$ converges to a point $x \in X$,
and $\{y_n\}$ converges to a point $y \in Y$.
So we can find an N_1 s.t. $d(x_n, x) < \frac{\varepsilon}{2}$ whenever $n \geq N_1$ and
an N_2 s.t. $d(y_n, y) < \frac{\varepsilon}{2}$ whenever $n \geq N_2$. Let $N' = \max(N_1, N_2)$.
Then we have that $d((x_n, y_n), (x, y)) = d(x_n, x) + d(y_n, y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$
whenever $n \geq N'$, so $\{(x_n, y_n)\}$ must converge to (x, y). ∎

③ As X is complete, need to show that $\{a_n\}$ is a Cauchy-sequence.
Choose any $\varepsilon > 0$. As $\operatorname{diam}(A_n) \to 0$, can find $N \in \mathbb{N}$ s.t. $\operatorname{diam}(A_n) < \varepsilon$
whenever $n \geq N$. Then, when $m, n \geq N$, we have $a_m \in A_m$, $a_n \in A_n$.
Can assume $m \geq n$ so we have $A_m \subseteq A_n$, and $d(x_m, x_n) \leq \operatorname{diam} A_n < \varepsilon$.
So $d(x_n, x_m) < \varepsilon$ whenever $m, n \geq N \Rightarrow a_n$ is a Cauchy-sequence,
and must therefore converge.

∎

(5) Want to use Banach's Fixed Point Theorem, need to show that f is a contraction.

Choose $x, y \in [0,1]$, $x<y$. By the Mean Value Theorem, there exists a $c \in (x,y)$ s.t. $f'(c) = \dfrac{f(y)-f(x)}{y-x}$.

So we have $|f'(c)| = \dfrac{|f(y)-f(x)|}{|y-x|} \Rightarrow |y-x||f'(c)| = |f(y)-f(x)|$.

$\Rightarrow d(f(x),f(y)) = |f'(c)| d(x,y) < S \cdot d(x,y) \Rightarrow f$ is a contraction.

Then, by Banach's Fixed point theorem, there exists exactly one $a \in [0,1]$ s.t. $f(a)=a$.

(6) Let us assume the world is complete. (If you study physics, try to prove this!) (If you actually prove this, mention me in the Nobel Prize speech).

Let f be the function that sends a point in the world to its corresponding point on the map.

This is clearly a contraction, unless you have a 1:1 map of the area, and that seems unwieldy. By Banach's Fixed Point Theorem, there is exactly one point that is sent to itself by f, and this is the point that is vertically above itself.

(7) As f^{on} is a contraction, it has a unique fixed point a s.t. $f^{on}(a)=a$. But then we have $f(a) = f(f^{on}(a)) = f^{o(n+1)}(a) = f^{on}(f(a))$, so $f^{on}(f(a)) = f(a)$, and $f(a)$ is also a fixed point for f^{on}.

But the fixed point was supposed to be unique, so we must have $f(a)=a$, which shows that f has a as a fixed point as well.

And it must be unique, for if $f(b)=b$, then $f^{on}(b)=b$ as well, implying $b=a$.

8

Let $\{x_n\}$ be a Cauchy-sequence. Choose an $\varepsilon > 0$. Will find an $x \in X$ s.t. $d(x_n, x) < \varepsilon$ whenever $n \geq N$, for some $N \in \mathbb{N}$.

As $\{x_n\}$ is Cauchy, we can find an N_1 s.t. $d(x_n, x_m) < \frac{\varepsilon}{3}$ when $n, m \geq N_1$.

As D is dense in X, we can for each x_n find a $y_n \in D$ s.t. $d(x_n, y_n) < \frac{\varepsilon}{3}$. Then we have that

$$d(y_n, y_m) \leq d(y_n, x_n) + d(x_n, x_m) + d(x_m, y_m) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

whenever $n, m \geq N_1$, so $\{y_n\}$ is a Cauchy sequence.

By assumption, there exists an $x \in X$ s.t. $y_n \to x$.

So we can find an $N \in \mathbb{N}$ s.t. $d(y_n, x) < \frac{2\varepsilon}{3}$ whenever $n \geq N$, and then we have

$$d(x_n, x) \leq d(x_n, y_n) + d(y_n, x) < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon$$ whenever $n \geq N$, as needed.

So all Cauchy-sequences converge, and X is complete.

3.5

① As X is closed in itself, by Thm 3.5.13 we need to prove that

X is finite $\Leftrightarrow$ X is totally bounded.

If X is finite, $X=\{x_1,\dots,x_n\}$, you can choose any $\varepsilon>0$,
and we have $X=B(x_1,\varepsilon)\cup\dots\cup B(x_n,\varepsilon)$, so X is totally bounded.

If X is totally bounded, we can, for $\varepsilon=\frac{1}{2}$, find points
$x_1,\dots,x_n$ s.t. $X=B(x_1,\frac{1}{2})\cup\dots\cup B(x_n,\frac{1}{2})$.
But $B(x_i,\frac{1}{2})=\{x_i\}$, so we have $X=\{x_1,\dots,x_n\}$, and X is finite.

② Important to note: If $\{y_k\}=\{x_{n_k}\}$ is a subsequence of $\{x_n\}$,
then $k\le n_k$.

Let $\{x_n\}$ be a sequence that converges to a.
Then, for any $\varepsilon>0$, we have an $N\in\mathbb{N}$ s.t. $d(x_n,a)<\varepsilon$ whenever $n\ge N$.
Now, we have $k\le n_k$, so $k\ge N \Rightarrow n_k\ge N$, $\Rightarrow d(x_{n_k},a)<\varepsilon$.
So $d(x_{n_k},a)=d(y_k,a)<\varepsilon$ whenever $k\ge N$, i.e. $y_k\to a$.

③ Let us follow (copy) the proof in 3.5.10.
Let $m=\inf\{f(x)\mid x\in K\}$. Possibly, $m=-\infty$.
Choose a sequence $\{x_n\}$ in K s.t. $\lim_{n\to\infty} f(x_n)=m$.
Since K is compact, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging
to a point $d\in K$. Then, on the one hand, $\lim_{k\to\infty} f(x_{n_k})=m$, and
on the other $\lim_{k\to\infty} f(x_{n_k})=f(d)$ according to Prop. 3.2.9.

Hence $f(d)=m$, and since $m=\inf\{f(x)\mid x\in K\}$, we see that
d is a minimum point for f on K.

(4)

a) A is bounded, so there exists an M s.t. $d(a,b) \leq M$ for all $a,b \in A$. Then, specifically $d(a,c) \leq M$ for all $a \in A$, so we let $M_c = M$.

b) Choose two points $a, b \in A$. We have $d(a,c) \leq M$, $d(c,b) \leq M$, so by the triangle inequality $d(a,b) \leq d(a,c) + d(c,b) \leq M + M = 2M$. This is true for any $a,b \in A$, so A is bounded.

(5) If we just choose an $\varepsilon > 0$, and look at K_ε, we see that $f(x) < \varepsilon$ for all $x \notin K_\varepsilon$, and there exists a point $c \in K_\varepsilon$ s.t. $f(c) \geq f(x)$ for all $x \in K_\varepsilon$. <u>However</u> we do not know if $f(c) \geq \varepsilon$, and if $f(c) < \varepsilon$ we might find $x \notin K_\varepsilon$ with $f(x) > f(c)$. Let us choose ε wisely.

Choose any point $z \in X$. Let $\varepsilon = f(z)$. Then, as $f(x) < \varepsilon = f(z)$ for all $x \notin K_{f(z)}$, we have more to work with.

As $K_{f(z)}$ is compact, there exists a $c \in K_{f(z)}$ with $f(c) \geq f(x)$ for all $x \in K_\varepsilon$. As $f(z) > f(x)$ for all $x \notin K_{f(z)}$, we must have $z \in K_{f(z)}$. So $f(c) \geq f(z) > f(x)$ for all $x \in K_{f(z)}$.

Now we have both $f(c) \geq f(x)$ for all $x \in K_{f(z)}$ and $f(c) \geq f(x)$ for all $x \notin K_{f(z)}$, so

$f(c) \geq f(x)$ for all $x \in X$, and we have a maximum point.

e) As X is compact, we attain a minimum, i.e. there exists a $d \in X$ s.t. $f(d) \leq f(x)$ for all $x \in X$. As $f(x) > 0$ for all $x \in X$, $f(d) > 0$. Choose $a = \frac{f(d)}{2} < f(d)$, and we have $a < f(d) \leq f(x)$ for all $x \in X$.

7

K is compact $\Rightarrow$ K is closed.

K is closed $\Rightarrow$ $f^{-1}(K)$ is closed by Prop. 3.3.10.

Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 0$. We have that $\{0\}$ is closed and bounded $\Rightarrow$ $\{0\}$ is compact.

But $f^{-1}(\{0\}) = \mathbb{R}$ is not bounded, so not compact.

Tidbit: If f is a function where $f^{-1}(K)$ is compact for any compact K, we call f a proper function.