1 3.5)

(5) $f: X \to [0,\infty)$ cont. There exists compact K_ε s.t. $f(x) < \varepsilon$ for $x \notin K_\varepsilon$.
Show f has a max point.
See solution to last week's exercises.

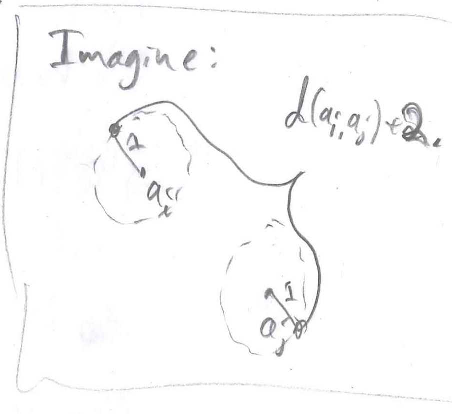
(6) $f: X \to \mathbb{R}$, X compact. Show if $f(x) > 0$ for all $x \in X$, there exists $a > 0$ s.t. $f(x) > a$ for all x.
See solution to last week's exercises.

(7) $f: X \to Y$, $K \subseteq Y$ compact. Show $f^{-1}(K)$ closed, not necessarily compact.
See solution to last week's exercises.

(8) Let A be a totally bounded subset of X.
Show that A is bounded, i.e. find an $M > 0$ s.t.
$d(a,b) \leq M$ for all $a,b \in A$.

As A is totally bounded, we can choose $\varepsilon = 1$, and find points $a_1, \dots, a_n$ s.t. $\bigcup_{i=1}^{n} B(a_i, 1) \supseteq A$.

Choose $M = \max_{i,j} \{d(a_i, a_j)\} + 2$.



Then, for any $a, b \in A$, we have
$a \in B(a_k, 1)$, $b \in B(a_l, 1)$, for some k, l.

so
$$\begin{aligned} d(a,b) &\leq d(a,a_k) + d(a_k, a_l) + d(a_l, b) \\ &< 1 + d(a_k, a_l) + 1 \\ &\leq 1 + \max_{i,j}\{d(a_i,a_j)\} + 1 \\ &= M. \end{aligned}$$

We have shown $d(a,b) \leq M$ for all $a, b \in A$, so A is bounded.

⑧ cont.

Any infinite set with the discrete metric is not totally bounded by exer. 3.5.1. But $d(a,b) \leq 1$ for a,b in the set, so it is bounded. Hence bounded but not totally bounded.

⑪ $\{K_n\}$ is a sequence of non-empty, compact subsets of X s.t. $K_n \supseteq K_{n+1}$. Will prove $\bigcap_{n\in\mathbb{N}} K_n$ is non-empty.

Let us create a sequence. Let x_1 be any point in K_1, x_2 any point in K_2 etc. So $x_n \in K_n$.

Then, as $K_n \subseteq K_1$ for all n, $\{x_n\}$ is a sequence contained in K_1. As K_1 is compact, there must exist a convergent subsequence $\{y_k\} = \{x_{n_k}\}$ converging to a $y \in K_1$.

But now we have that $\{y_k\} \cap K_2$ is a subsequence of $\{y_k\}$, and must converge to y as well. As K_2 is closed, we have $y \in K_2$. In general:

$\{y_k\} \cap K_n$ is a subsequence of $\{y_k\}$ (only a finite number (max $n-1$) elements are removed from $\{y_k\}$) that is contained in K_n, which converges to y.

As K_n is closed, we have $y \in K_n$ as well.

Now, as $y \in K_n$ for all n, we have $y \in \bigcap_{n\in\mathbb{N}} K_n$, so it is not empty. ☒

<u>Trivia</u>: The same is <u>not</u> true if we just assume K_n closed and bounded.

E.g. if we have $\mathbb{N}$ with the discrete metric, we can let $K_n = [n, \infty)$, which is closed and bounded, but

$\bigcap_{n\in\mathbb{N}} K_n = \emptyset$.

⑫. Let's go Theorem-hunting!

By Prop 3.3.10, a function f is continuous if and only if, whenever F is closed, $f^{-1}(F)$ is closed.

So to prove that f^{-1} is continuous, we can prove that if F closed in X, then $f(F)$ is closed in Y.

By Prop 3.5.8, a closed subset of a compact set is compact. So as X is compact, any closed $F \subseteq X$ must also be compact.

By Prop. 3.5.9, if $f: X \to Y$ is continuous, we have that $f(K)$ must be compact for any compact K.

Lastly, by Prop. 3.5.4, any compact set is closed.

So: If F is a closed set in X, F must be compact (3.5.8). Therefore $f(F)$ is compact (3.5.9), i.e. closed (3.5.4). And as $f(F)$ is closed for all F closed, f^{-1} must be continuous, by 3.3.10. $\blacksquare$

(14) First, a quick proof:

Let $\{x_n\}$ and $\{y_n\}$ be sequences converging to x and y resp.
Then $d(x_n, y_n) \to d(x,y)$.

Pf: Choose any $\varepsilon > 0$. As $x_n \to x$, we can find $N_1 \in \mathbb{N}$ s.t. $d(x_n, x) < \frac{\varepsilon}{2}$ when $n \geq N_1$. As $y_n \to y$, we can find $N_2 \in \mathbb{N}$ s.t. $d(y_n, y) < \frac{\varepsilon}{2}$ when $n \geq N_2$. Let $N = \max\{N_1, N_2\}$.

Then: $d(x,y) \leq d(x,x_n) + d(x_n, y_n) + d(y_n, y)$

$\Rightarrow d(x,y) - d(x_n, y_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

$d(x_n, y_n) \leq d(x_n, x) + d(x,y) + d(y, y_n)$

$\Rightarrow d(x_n, y_n) - d(x,y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

So: $|d(x_n, y_n) - d(x,y)| < \varepsilon$ whenever $n \geq N$

$\Rightarrow d(x_n, y_n) \to d(x,y)$.

---

a) Now, let x_n be any sequence converging to an $x \in X$.
As f is continuous, $f(x_n)$ converges to $f(x)$, by Prop 3.2.5.
We have $g(x_n) = d(x_n, f(x_n))$, which converges to $d(x, f(x))$ by.
So by Prop 3.2.5, g is continuous at x. This works for all x, so g is continuous.

Now, by the Extreme Value Theorem, we have that g must have a minimum point, as X is compact.

b) Let c be the minimum point for g, i.e. $g(c) \leq g(x)$ for all $x \in X$.
Assume $c \neq f(c)$. Then we have $g(f(c)) = d(f(c), f(f(c))) < d(c, f(c)) = g(c)$, impossible. So must have $f(c) = c$, and we have a fixed point.

To show it's unique: Assume we have two fixed points, $c \neq d$.
Then $d(c,d) > d(f(c), f(d)) = d(c,d)$, impossible. So must have $c = d$, unique fixed point.

3.6

① $[0,1]$ is closed and bounded in $\mathbb{R}$, and is therefore compact.
As $[0,1]$ is compact, it has the Open Covering Property, so we can find a finite subset $I_1, \dots, I_n$ that contains $[0,1]$. ☒

② This exercise is the same as 3.5.11, but we can come up with a shorter proof now!
Let $\mathcal{F} = \{K_n\}_{n=1}^{\infty}$. Then $\mathcal{F}$ has the finite intersection property over K_1, as

$$K_1 \cap F_1 \cap \dots \cap F_n = K_1 \cap K_{i_1} \cap K_{i_2} \cap \dots \cap K_{i_n} = K_N \text{ where } N = \max\{i_1, \dots, i_n\}, \neq \emptyset.$$

As K_1 is compact, we have from 3.6.5 that

$$K_1 \cap \bigcap_{F \in \mathcal{F}} F = K_1 \cap \bigcap_{n \in \mathbb{N}} K_n = \bigcap_{n \in \mathbb{N}} K_n \neq \emptyset.$$ ☒

③ Choose any open covering of $f(K)$, $\mathcal{U} = \{U_i\}$, $\bigcup_i U_i \supseteq f(K)$.
As $f^{-1}(f(K)) \supseteq K$ and $f^{-1}(\bigcup_i U_i) = \bigcup_i f^{-1}(U_i)$, we have that $K \subseteq \bigcup_i f^{-1}(U_i)$, and as f is continuous, $f^{-1}(U_i)$ is open.
So this is an open covering of K. As K is compact, there exists a finite subcovering $U_1, \dots, U_n$ s.t.

$$K \in f^{-1}(U_1) \cup f^{-1}(U_2) \cup \dots \cup f^{-1}(U_n).$$

Now, for any $y \in f(K)$, there exists an $x \in K$ s.t. $f(x) = y$.
Then we have $x \in f^{-1}(U_j)$ for some $1 \le j \le n$, i.e. $f(x) = y \in U_j$.
So $U_1 \cup U_2 \cup \dots \cup U_n \supseteq f(K)$, and we have a finite subcovering of $\mathcal{U}$.
Therefore, $f(K)$ is compact. ☒

④ Let $\mathcal{U} = \{U_i\}$ be any open covering of $K_1 \cup K_2 \cup \dots \cup K_n$.

As K_1 is compact and $K_1 \subseteq K_1 \cup \dots \cup K_n \subseteq \bigcup_i U_i$, there exists a finite subcovering $U_1^1, U_2^1, \dots, U_m^1$ with $K_1 \subseteq U_1^1 \cup U_2^1 \cup \dots \cup U_m^1$.

The same is true for $K_2, K_3, \dots, K_n$ so in general, for $i = 1, \dots, n$, we have $K_i \subseteq U_1^i \cup U_2^i \cup \dots \cup U_{m_i}^i$

We can now look at the collection $\mathcal{U}' = \{ U_j^i : i = 1, 2, \dots, n,\ j = 1, 2, \dots, m_i \}$.

This collection is finite, and we have

$$K_i \subseteq \bigcup_{U \in \mathcal{U}'} U \quad \text{for all } i, \text{ so } K_1 \cup \dots \cup K_n \subseteq \bigcup_{U \in \mathcal{U}'} U.$$

This is therefore a finite subcovering and $K_1 \cup \dots \cup K_n$ is compact.

$\blacksquare$

4.1

① Will start by assuming that $f(x)=x^2$ is uniformly continuous, and get a contradiction.

Given an $\varepsilon > 0$, we assume there exists a $\delta>0$ s.t. $|f(x)-f(y)|<\varepsilon$ whenever $|x-y|<\delta$, for all $x, y \in \mathbb{R}$.

Now, let $x=\frac{\varepsilon}{\delta}$, $y=\frac{\varepsilon}{\delta}+\frac{\delta}{2}$.

What you really do here: Let $y=x+\frac{\delta}{2}$ s.t. $|x-y|<\delta$. Then try to solve $|f(x)-f(y)|>\varepsilon$. $x=\frac{\varepsilon}{\delta}$ is just a nice choice that solves this.

Then $|x-y|=|-\frac{\delta}{2}|<\delta$, so we should have $|f(x)-f(y)|<\varepsilon$.

But $|f(x)-f(y)|=|x^2-y^2|=|x+y||x-y|$

$$=\left|\frac{2\varepsilon}{\delta}+\frac{\delta}{2}\right|\left|\frac{\delta}{2}\right|=\varepsilon+\frac{\delta^2}{4}>\varepsilon,$$ a contradiction.

So $f(x)=x^2$ cannot be uniformly continuous.

② Will start by assuming that $f(x)=x^2$ is uniformly continuous, and get a contradiction.

Given an $\varepsilon>0$, we assume there exists a $\delta>0$ s.t. $|f(x)-f(y)|<\varepsilon$ whenever $|x-y|<\delta$, for all $x,y\in\mathbb{R}$.

Could start with an x, $y=x+\frac{\delta}{2}$, but will be easier to work with if we assume $x<1$, $y=x(1+\delta)$. Then $|x-y|=|x|\delta<\delta$, and $\left|\frac{1}{x}-\frac{1}{y}\right|=\frac{1}{x}-\frac{1}{x(1+\delta)}=\frac{1+\delta-1}{x(1+\delta)}=\frac{\delta}{x(1+\delta)}$.

Want to find an x s.t. $\left|\frac{1}{x}-\frac{1}{y}\right|=\frac{\delta}{x(1+\delta)}\geq\varepsilon \Rightarrow x\leq\frac{\delta}{\varepsilon(1+\delta)}$,

So we choose $x=\min\{0.9, \frac{\delta}{\varepsilon(1+\delta)}\}$ (we needed $x<1$), and $y=x+\delta x$. Then $|x-y|=|x|\delta=x\delta<\delta$, and

$$\left|\frac{1}{x}-\frac{1}{y}\right|\geq\left|\frac{1}{\frac{\delta}{\varepsilon(1+\delta)}}-\frac{1}{\frac{\delta}{\varepsilon(1+\delta)}(1+\delta)}\right|=\left|\frac{\varepsilon(1+\delta)}{\delta}-\frac{\varepsilon}{\delta}\right|=\left|\frac{\varepsilon\delta}{\delta}\right|=\varepsilon,$$

a contradiction. So $f(x)=\frac{1}{x}$ cannot be uniformly continuous.

③ We want to show that given an $\varepsilon > 0$, we can find a $\delta > 0$ s.t. $d_Y(f(x), f(y)) < \varepsilon$ whenever $d_X(x,y) < \delta$, for all $f \in \mathcal{F}$.

As all $f \in \mathcal{F}$ are Lipschitz-continuous with the same Lipschitz-constant K, we have that $d_Y(f(x), f(y)) \leq K \cdot d_X(x,y) < K\delta$ whenever $d_X(x,y) < \delta$. So by choosing $\delta = \frac{\varepsilon}{K}$, we have that

$$d_Y(f(x), f(y)) < K\delta = K\frac{\varepsilon}{K} = \varepsilon$$ whenever $d_X(x,y) < \delta$, for all $f \in \mathcal{F}$,

and therefore $\mathcal{F}$ is equicontinuous.

④ If f' is bounded, we have an $M \in \mathbb{R}$ s.t. $|f'(x)| < M$ for all $x \in \mathbb{R}$.

For a given $x, y \in \mathbb{R}$, the Mean Value Thm tells us that there exists a $c \in (x,y)$ (or possibly (y,x)), s.t. $|f'(c)| = \frac{|f(x) - f(y)|}{|x-y|}$.

And as $|f'(c)| < M$, we have $\frac{|f(x)-f(y)|}{|x-y|} < M$ for all $x, y \in \mathbb{R}$,

So $|f(x) - f(y)| < M|x-y| \Rightarrow d_{\mathbb{R}}(f(x), f(y)) < M\delta$ whenever $d_{\mathbb{R}}(x,y)$.

Now, given an $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{M}$, and we have

$$d_{\mathbb{R}}(f(x), f(y)) < M\delta = \frac{M\varepsilon}{M} = \varepsilon$$ whenever $d_{\mathbb{R}}(x,y) < \delta$,

so f is uniformly continuous.