4.21

① For pointwise convergence, must show that given an x and $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ s.t. $|f_n(x)| < \varepsilon$ whenever $n \geq N$.

As $|f_n(x)| = \frac{|x|}{n}$, we want $\frac{|x|}{n} < \varepsilon \Rightarrow n > \frac{|x|}{\varepsilon}$. Choose $N = \lceil \frac{|x|}{\varepsilon} \rceil$, and we have what we want.

To show that this is not uniform convergence, can show that for an $\varepsilon > 0$, no matter the choice of N, I can find an x s.t. $|f_N(x)| > \varepsilon$. Choose $\varepsilon = 1$. Then, if given an N, I can choose an $x > N$. Then $f_N(x) = \frac{x}{N} > 1 = \varepsilon$, as wanted.

② For pointwise convergence, must show that given an x and $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ s.t. $|f_n(x)| < \varepsilon$ whenever $n \geq N$.

As $|f_n(x)| = x^n$, we want $x^n < \varepsilon \Rightarrow n > \frac{\ln \varepsilon}{\ln x}$. ($\ln x < 0$, so remember to change inequality)

Choose $N = \left\lceil \frac{\ln \varepsilon}{\ln x} \right\rceil$, and we have what we want.

To show that this is not uniform convergence, can show that for an $\varepsilon > 0$, no matter the choice of N, I can find an x s.t. $|f_N(x)| > \varepsilon$. Can choose ε any number less than 1, say $\varepsilon = \frac{1}{2}$. Then, if given an N, $\sqrt[N]{\frac{1}{2}} < 1$, and I can choose an $x \in (0,1)$, $x > \sqrt[N]{\frac{1}{2}}$. Thus, $f_N(x) = x^N > \left(\sqrt[N]{\frac{1}{2}}\right)^N = \frac{1}{2} = \varepsilon$, as wanted.

3

a) To show pointwise convergence, we must first figure out what it should converge to. As $\frac{x}{n} \to 0$, and $a^n \to 0$ if $a<1$, 0 is a good guess.

Given x and ε, want to find an N s.t. $e^{-x}\left(\frac{x}{n}\right)^{ne} < \varepsilon$, when $n \geq N$.

when $n > x$, $\frac{x}{n} < 1$, and then $\left(\frac{x}{n}\right)^{ne} < \frac{x}{n}$.

So if we choose n s.t. $\frac{x}{n} < \min(1, e^x\varepsilon)$, we have

$$e^{-x}\left(\frac{x}{n}\right)^{ne} < e^{-x}\frac{x}{n} < e^{-x}e^{x}\varepsilon = \varepsilon.$$

So choose $N > \frac{x}{\min(1, e^x\varepsilon)}$, and we have what we want.

b) As $x \to \infty$, $f_n(x) \to 0$, so to find the maximum of $f_n(x)$, we must differentiate.

$$f_n'(x) = -e^{-x}\left(\frac{x}{n}\right)^{ne} + e^{-x}\frac{ne}{n}\left(\frac{x}{n}\right)^{ne-1} = \left(\frac{x}{n}\right)^{ne-1}e^{-x}\left(e - \frac{x}{n}\right)$$

$$f_n'(x) = 0 \Rightarrow e - \frac{x}{n} = 0 \Rightarrow x = ne.$$

$$f(ne) = e^{-ne}\left(\frac{\cancel{n}e}{\cancel{x}}\right)^{ne} = e^{-ne}e^{ne} = 1.$$

So the maximum value of $f_n(x)$ is 1, and $f_n(x)$ cannot converge uniformly, as $\sup_{x\in[0,\infty)}\{|f_n(x)|\} = 1$, which does not go to 0 as $n \to \infty$.

④

Want to show that $\lim_{n\to\infty} n(x^{1/n}-1) = \ln x$.

Will use L'Hopital, differentiate with respect to n.

$$\lim_{n\to\infty} n(x^{1/n}-1) = \lim_{n\to\infty} \frac{x^{1/n}-1}{1/n} \overset{L'H}{=} \lim_{n\to\infty} \frac{-\cancel{\frac{1}{n^2}}\ln x \cdot x^{1/n}}{-\cancel{\frac{1}{n^2}}}$$

$$= \lim_{n\to\infty} x^{1/n}\ln x = \ln x \text{ as } x^{1/n}\to 1$$

when $n\to\infty$.

To show that the convergence is uniform, we must look at the difference $n(x^{1/n}-1)-\ln x$.

We see that when $x=1$, $n(1^{1/n}-1)-\ln(1) = n\cdot 0 - 0 = 0$

Let us look at the derivative.

$$\left(n(x^{1/n}-1)-\ln x\right)' = x^{1/n-1} - x^{-1}$$

When $x>1$, $x > x^{1-1/n} \Rightarrow \frac{1}{x^{1-1/n}} > \frac{1}{x} \Rightarrow x^{1/n-1} - x^{-1} > 0$.

When $x<1$, $x < x^{1-1/n} \Rightarrow \frac{1}{x^{1-1/n}} < \frac{1}{x} \Rightarrow x^{1/n-1} - x^{-1} < 0$.

And when $x=1$, $x^{1/n-1} - x^{-1} = 0$.

So $x=1$ is a minimum, and $n(x^{1/n}-1)-\ln x$ grows both when $x\to\infty$ and when $x\to 0$.

So $\max_{x\in[1/k,k]} \left(n(x^{1/n}-1)-\ln x\right)$ must be either $x=\frac{1}{k}$ or $x=k$, with a maximum value of either $n\left(\left(\frac{1}{k}\right)^{1/n}-1\right)+\ln k$ or $n(k^{1/n}-1)-\ln k$.

In either case, we have $\limsup_{x\in[\frac{1}{k},k]} \{|n(x^{1/n}-1)-\ln x|\} = 0$,

So we have uniform convergence by Prop. 4.2.3, and

$$\sup_{x\in(\frac{1}{k},k)} \{|n(x^{1/n}-1)-\ln x|\} \le \sup_{x\in[1/k,k]} \{n(x^{1/n}-1)-\ln x\}.$$

④ cont.

We will compute $\lim_{x\to 0} n(x^{1/n}-1)-\ln x$. If this is ∞, we have $\sup_{x\in(0,\infty)}\{|n(x^{1/n}-1)-\ln x|\} = \infty$ and this cannot go to 0 as $n\to\infty$, so this cannot be uniform convergence.

We have:

$$\lim_{n\to\infty} n(x^{1/n}-1)-\ln x = \lim_{n\to\infty} \ln\left(e^{n(x^{1/n}-1)-\ln x}\right)$$

$$= \lim_{n\to\infty} \ln\left(\frac{e^{n(x^{1/n}-1)}}{x}\right)$$

As $x\to 0$, $x^{1/n}\to 0$, so $e^{n(x^{1/n}-1)} \to e^{-1}$. $e^{-1}/x \to \infty$, so we have $\ln\left(\frac{e^{n(x^{1/n}-1)}}{x}\right) \to \infty$ as well, and therefore we do not have uniform convergence on $(0,\infty)$.

⑤ For any point $a\in\mathbb{R}$, we have a $k\in\mathbb{N}$ s.t. $a\in(-k,k)$. On $[-k,k]$ we have $f_n\to f$ uniformly, so by Prop. 4.2.4, f must be continuous on $[-k,k]$, specifically on a. As this is true for all $a\in\mathbb{R}$, f must be continuous.

⑥ As $f_n\to f$ uniformly, we have $\sup_{x\in X}\{|f_n(x)-f(x)|\}\to 0$ as $n\to\infty$

As $g_n\to g$ uniformly, we have $\sup_{x\in X}\{|g_n(x)-g(x)|\}\to 0$ as $n\to\infty$

So there exists an N s.t. $\sup_{x\in X}\{|f_n(x)-f(x)|\}<\frac{\varepsilon}{2}$ and $\sup_{x\in X}\{|g_n(x)-g(x)|\}<\frac{\varepsilon}{2}$ when $n\geq N$.

Therefore

$$\sup_{x\in X}\{|f_n(x)+g_n(x)-f(x)-g(x)|\} \leq \sup_{x\in X}\{|f_n(x)-f(x)|+|g_n(x)-g(x)|\}$$

$$\leq \sup_{x\in X}\{|f_n(x)-f(x)|\}+\sup_{x\in X}\{|g_n(x)-g(x)|\}$$

$$< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

whenever $n\geq N$. So $\sup_{x\in X}\{|f_n(x)+g_n(x)-(f(x)+g(x))|\}\to 0$ as $n\to\infty$, which means that f_n+g_n converges to $f+g$ uniformly.

9

Want to show that we can find an $N \in \mathbb{N}$ s.t.

$$d(f_n(x_n), f(x)) < \varepsilon \text{ whenever } n \geq N.$$

As f is continuous, (by Prop. 4.24), we can find a δ s.t. $d(f(x_n), f(x)) < \frac{\varepsilon}{2}$ whenever $d(x_n, x) < \delta$.

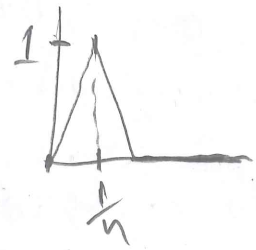
As f_n converges uniformly to f, we can find an $N_1 \in \mathbb{N}$ s.t. $d(f_n(x_n), f(x_n)) < \frac{\varepsilon}{2}$ whenever $n \geq N_1$ (True for all $x \in X$, and therefore specifically for x_n).

As $x_n \to x$, we can find an $N_2 \in \mathbb{N}$ s.t. $d(x_n, x) < \delta$ whenever $n \geq N_2$.

So, with these values, we have that if $n \geq N = \max(N_1, N_2)$, then $d(x_n, x) < \delta \Rightarrow d(f(x_n), f(x)) < \frac{\varepsilon}{2}$ and $d(f_n(x_n), f(x_n)) < \frac{\varepsilon}{2}$.

so $d(f_n(x_n), f(x)) \leq d(f_n(x_n), f(x_n)) + d(f(x_n), f(x)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Let $f_n(x)$ be given by this graph:



Let $x_n = \frac{1}{n}$. Then $x_n \to 0$, $f(0) = 0$, but $f_n(x_n) = 1$ for all n, so $f_n(x_n) \to 1$, not 0.

(12)

a) As $d(g_n(x), 0) = |g_n(x)| = |f(x) - f_n(x)| = d(f(x), f_n(x))$,
if there exists an $N \in \mathbb{N}$ s.t. $d(g_n(x), 0) < \varepsilon$ for all $x \in X$, when $n \geq N$,
then $d(f(x), f_n(x)) < \varepsilon$ for all $x \in X$, when $n \geq N$ as well,
and we have uniform convergence.

b) g_n does not converge uniformly to 0
$\Updownarrow$
There exists an ε s.t. for all $N \in \mathbb{N}$, you can find an x_N s.t.
$|g_n(x_N)| \geq \varepsilon$ for some $n \geq N$.
As $g_n(x) \geq 0$ for all x, we do not need the absolute value, and
as $g_n(x_N)$ is decreasing, $g_n(x_N) \geq \varepsilon$ for some $n \geq N \Rightarrow g_N(x_N) \geq \varepsilon$.
So, as there exist an x_N for each N s.t. $g_N(x_N) \geq \varepsilon$ we have what we want.

c) As X is compact, every sequence must have a convergent subsequence.
So we have a subsequence converging to a point $a \in X$.

d) As $g_n(a)$ converges to 0, we can find an $N \in \mathbb{N}$, s.t. $g_n(a) < \frac{\varepsilon}{2}$
whenever $n \geq N$.
As g_N is continuous, (it is the difference of two continuous functions),
there exists an r s.t. $d(g_N(x), g_N(a)) < \frac{\varepsilon}{2}$ whenever $d(x, a) < r$.
So if $x \in B(a,r) \Rightarrow d(x, a) < r \Rightarrow d(g_N(x), g_N(a)) < \frac{\varepsilon}{2}$, and $g_N(a) < \frac{\varepsilon}{2}$.
This gives us $g_N(x) = d(g_N(x), 0) \leq d(g_N(x), g_N(a)) + d(g_N(a), 0) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$,
when $x \in B(a,r)$, as wanted.

e) As $x_{n_k} \to a$, there exists a $K \in \mathbb{N}$ s.t. $d(x_{n_k}, a) < r$ whenever $k \geq K$.
Then $x_{n_K} = x_m \in B(a,r)$. We can choose K large enough s.t. $m \geq N$, where
N is chosen as in d).
So $g_N(x_m) < \varepsilon$ by d), and $g_m(x_m) \geq \varepsilon$ by b).
But g_n decreases to 0, so $g_m(x_m) \leq g_N(x_m) \Rightarrow \varepsilon < \varepsilon$, a contradiction.

4.3

① We have $\left|\frac{\cos nx}{n^2+1}\right| \le \frac{1}{n^2+1} \le \frac{1}{n^2} = M_n.$

As $\sum_{n=0}^{\infty} M_n = \sum_{n=0}^{\infty} \frac{1}{n^2}$ converges, $\sum_{n=0}^{\infty} \frac{\cos nx}{n^2+1}$ converges uniformly by Weierstrass M-test.

② No. Let us look at $\frac{e^{-x}}{(1-e^{-x})^2} - \sum_{n=1}^{N} n e^{-nx}$ as $x \to 0$.

We get $\lim_{x\to 0} \frac{e^{-x}}{(1-e^{-x})^2} - \sum_{n=1}^{N} n e^{-nx} = \frac{1}{(1-1)^2} - \sum_{n=1}^{N} n = \infty.$

So no matter which N we choose, we can find an x sufficiently close to 0 s.t. $d\left(\frac{e^{-x}}{(1-e^{-x})^2}, \sum_{n=1}^{N} ne^{-nx}\right) \ge \varepsilon$, and the convergence is therefore not uniform on $(0,\infty)$.

③ Want to compute $\lim_{n\to\infty} nx(1-x^2)^n$. As $x \in [0,1]$, let us first check $x=0, x=1$. If $x=0$, $nx(1-x^2)^n = n\cdot 0\cdot(1-0^2)^n = 0$

If $x=1$, $nx(1-x^2)^n = n\cdot 1(1-1)^n = 0$

If $x \in (0,1)$, $x^2 \in (0,1)$ as well, so $1-x^2 \in (0,1) \Rightarrow (1-x^2)^n \to 0$.

Will use L'Hopital.

$$\lim_{n\to\infty} nx(1-x^2)^n = \lim_{n\to\infty} \frac{nx}{(1-x^2)^{-n}} \overset{L'H}{=} \lim_{n\to\infty} \frac{x}{-\ln(1-x^2)(1-x^2)^{-n}}$$

$$= \lim_{n\to\infty} \frac{-x(1-x^2)^n}{\ln(1-x^2)} = \frac{-x\cdot 0}{\ln(1-x^2)} = 0.$$

Now to compute $\int_0^1 f_n(x)dx$.

$u = 1-x^2 \quad \frac{du}{dx} = -2x \quad du = -2x\,dx$

$$\int_0^1 f_n(x)dx = \int_0^1 nx(1-x^2)^n dx = \int_{1-0^2}^{1-1^2} -\frac{n}{2} u^n du$$

$$= \left[-\frac{n}{2(n+1)} u^{n+1}\right]_1^0 = 0 - \left(-\frac{n}{2(n+1)}\right) = \frac{n}{2(n+1)} \to \frac{1}{2} \text{ as } n\to\infty.$$

5) a)

We have $\left|\frac{\cos\frac{x}{n}}{n^2}\right| \le \frac{1}{n^2}$ on $\mathbb{R}$, and $\sum \frac{1}{n^2}$ converges,

so by Weierstrass' M-test, $\sum_{n=1}^{\infty} \frac{\cos\frac{x}{n}}{n^2}$ must converge uniformly.

b) By Corr 4.3.3, Then $\sum_{n=1}^{\infty} \int_0^x \frac{\cos\frac{t}{n}}{n^2} dt$ converges uniformly

on any interval $[a,b]$ containing 0.

$$\int_0^x \frac{\cos\frac{t}{n}}{n^2} dt = \left[\frac{n\sin\frac{t}{n}}{n^2}\right]_0^x = \frac{\sin\frac{x}{n}}{n} - 0 = \frac{\sin\frac{x}{n}}{n}.$$

So $\sum_{n=1}^{\infty} \frac{\sin\frac{x}{n}}{n}$ converges uniformly on all $[-k,k]$, and is therefore

continuous on all $[-k,k]$. And as any $a \in \mathbb{R}$ is contained in an interval $[-k,k]$, we have that $\sum_{n=1}^{\infty} \frac{\sin\frac{x}{n}}{n}$ is continuous at a for all $a \in \mathbb{R}$.

$\Rightarrow \sum_{n=1}^{\infty} \frac{\sin\frac{x}{n}}{n}$ is continuous.

By Corr 4.3.3, $f'(x) = \sum_{n=1}^{\infty}\left(\frac{\sin\frac{x}{n}}{n}\right)' = \sum_{n=1}^{\infty} \frac{\frac{1}{n}\cos\frac{x}{n}}{n} = \sum_{n=1}^{\infty} \frac{\cos\frac{x}{n}}{n^2}$,

as wanted.

We can use Corr 4.3.3 on any interval $[-k,k]$, as we know that $\sum_{n=1}^{\infty} \frac{\cos\frac{x}{n}}{n^2}$ converges uniformly and $\sum_{n=1}^{\infty} \frac{\sin\frac{0}{n}}{n} = 0$

converges. Any $x \in \mathbb{R}$ is contained in a $[-k,k]$, so we get our derivative.

2.21

4

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup_{k\geq n} \{a_k\} = \lim_{n\to\infty} \sup_{k\geq n} \{(-1)^k\}$$

$$= \lim_{n\to\infty} 1 = 1$$

as $(-1)^k = \pm 1$, and for any n, can find k s.t. $a_k = 1$, $k \geq n$.

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf_{k\geq n} \{a_k\} = \lim_{n\to\infty} \inf_{k\geq n} \{(-1)^k\}$$

$$= \lim_{n\to\infty} -1 = -1$$

as $(-1)^k = \pm 1$, and for any n, can find k s.t. $a_k = -1$, $k \geq n$.

7

$$\limsup_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} \sup_{k\geq n} \{a_k + b_k\}$$

$$\leq \lim_{n\to\infty} \left(\sup_{k\geq n} \{a_k\} + \sup_{k\geq n} \{b_k\} \right)$$

as $\sup\{x+y\} \leq \sup\{x\} + \sup\{y\}$. The maximal value of the sum must be less than (or eq.) the sum of the maximal values.

$$= \lim_{n\to\infty} \sup_{k\geq n} \{a_k\} + \lim_{n\to\infty} \sup_{k\geq n} \{b_k\}$$

$$= \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

as wanted.

$$\liminf_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} \inf_{k\geq n} \{a_k + b_k\}$$

$$\geq \lim_{n\to\infty} \left(\inf_{k\geq n} \{a_k\} + \inf_{k\geq n} \{b_k\} \right)$$

as $\inf\{x+y\} \geq \inf\{x\} + \inf\{y\}$. The minimal value of the sum must be greater than or equal to the sum of the minimal values.

$$= \lim_{n\to\infty} \inf_{k\geq n} \{a_k\} + \lim_{n\to\infty} \inf_{k\geq n} \{b_k\}$$

$$= \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$$

As wanted.

7 cont. Let $a_n=(-1)^n$, $b_n=(-1)^{n+1}$.

Then $a_n+b_n=(-1)^n+(-1)^{n+1}=(-1)^n(1-1)=0$.

So $\limsup\limits_{n\to\infty}(a_n+b_n)=\limsup\limits_{n\to\infty} 0=0\le \limsup\limits_{n\to\infty} a_n+\limsup\limits_{n\to\infty} b_n=1+1=2$,

and $0\neq 2$.

$\liminf\limits_{n\to\infty}(a_n+b_n)=\liminf\limits_{n\to\infty} 0\ge \liminf\limits_{n\to\infty} a_n+\liminf\limits_{n\to\infty} b_n=-1+-1=-2$

and $0\neq -2$.

If a_n, b_n are two <u>positive</u> sequences, we have $\sup\limits_n\{a_nb_n\}\le \sup\limits_n\{a_n\}\sup\limits_n\{b_n\}$

and $\inf\limits_n\{a_nb_n\}\ge \inf\limits_n\{a_n\}\inf\limits_n\{b_n\}$.

So:
$$\begin{aligned}\limsup_{n\to\infty}(a_nb_n)&=\lim_{n\to\infty}\sup_{k\ge n}\{a_kb_k\}\\
&\le \lim_{n\to\infty}\left(\sup_{k\ge n}\{a_k\}\sup_{k\ge n}\{b_k\}\right)\\
&=\left(\lim_{n\to\infty}\sup_{k\ge n}\{a_k\}\right)\left(\lim_{n\to\infty}\sup_{k\ge n}\{b_k\}\right)\\
&=\left(\limsup_{n\to\infty} a_n\right)\left(\limsup_{n\to\infty} b_n\right),\end{aligned}$$

and
$$\begin{aligned}\liminf_{n\to\infty}(a_nb_n)&=\lim_{n\to\infty}\inf_{k\ge n}\{a_kb_k\}\\
&\ge \lim_{n\to\infty}\left(\inf_{k\ge n}\{a_k\}\inf_{k\ge n}\{b_k\}\right)\\
&=\left(\lim_{n\to\infty}\inf_{k\ge n}\{a_k\}\right)\left(\lim_{n\to\infty}\inf_{k\ge n}\{b_k\}\right)\\
&=\left(\liminf_{n\to\infty} a_n\right)\left(\liminf_{n\to\infty} b_n\right)\end{aligned}$$

$\square$

4.4

① Radius of convergence 0:

As we want $R=0$, and $\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}$,

we want $\limsup_{n\to\infty} |a_n|^{1/n} = \infty$.

We can choose $a_n = n^n$. Then $|a_n|^{1/n} = (n^n)^{1/n} = n$, so

$\limsup_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} n = \infty$.

Therefore:

$\sum_{n=0}^{\infty} n^n x^n$ has radius of convergence 0.

Radius of convergence 1:

Now we need $\limsup_{n\to\infty} |a_n|^{1/n} = 1$. If we let $a_n = 1$,

then $|a_n|^{1/n} = 1^{1/n} = 1$, so $\limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{n\to\infty} 1 = 1$.

Therefore:

$\sum_{n=0}^{\infty} x^n$ has radius of convergence 1.

Indeed, this is a geometric series, which converge if $|x|<1$.

Radius of convergence 2:

We want $\limsup_{n\to\infty} |a_n|^{1/n} = \frac{1}{2}$. Let $a_n = \frac{1}{2^n}$.

Then $|a_n|^{1/n} = \left(\frac{1}{2^n}\right)^{1/n} = \frac{1}{2}$, so $\limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{n\to\infty} \frac{1}{2} = \frac{1}{2}$.

Therefore:

$\sum_{n=0}^{\infty} \frac{1}{2^n} x^n$ has radius of convergence 2.

This is also a geometric series with quotient $\frac{x}{2}$, converging when $\frac{|x|}{2} < 1 \Rightarrow |x| < 2$.

① cont.

Radius of convergence ∞.

Easiest choice: Let $a_n = 0$. Then $\sum_{n=0}^{\infty} a_n x^n = 0$ for all x, so this converges for all x.

More exiting choice:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \quad \text{This has } \frac{1}{R} = \limsup_{n\to\infty} \left|\frac{1}{n!}\right|^{1/n} = \limsup_{n\to\infty} \frac{1}{n!^{1/n}}$$

We have
$$\ln(n!^{1/n}) = \frac{1}{n}\ln n! = \frac{\ln n}{n} + \frac{\ln(n-1)}{n} + \dots + \frac{\ln 1}{n}$$
$$\geq \frac{1}{n}\int_1^n \ln x\,dx$$
$$= \frac{1}{n}\Big[x(\ln x - 1)\Big]_1^n$$
$$= \frac{1}{n}\Big(n(\ln n - 1) - 1(\ln 1 - 1)\Big)$$
$$= \ln n - 1 + \frac{1}{n}$$

And as $n\to\infty$ $\ln n - 1 + \frac{1}{n} \to \infty$, so:

$$\limsup_{n\to\infty} \left|\frac{1}{n!}\right|^{1/n} = \lim_{n\to\infty} e^{\ln\left(\frac{1}{n!^{1/n}}\right)} = \lim_{n\to\infty} e^{-\ln(n!)^{1/n}}$$
$$\leq \lim_{n\to\infty} e^{-\ln n + 1 - \frac{1}{n}} = e^{-\infty} = 0.$$

So $\frac{1}{R} \leq 0$, but cannot be negative.

Therefore $\frac{1}{R} = 0 \Rightarrow R = \infty$.

There are many other choices as well.

③

P is a polynomial, $P(n) = c_k n^k + c_{k-1} n^{k-1} + \dots + c_0$

Compute $\lim_{n\to\infty} |P(n)|^{1/n}$.

Let us take the logarithm: $\ln |P(n)|^{1/n} = \frac{\ln |P(n)|}{n}$

We can assume n is so large that we have passed all zeros of $P(n)$.
Then $\ln(0)$ is not a problem, and we will either have $P(n) > 0$ for all large enough n, or $P(n) < 0$ for all large enough n.

Assume $P(n) > 0$ (otherwise just look at $Q(n) = -P(n)$, $\sqrt[n]{|P(n)|} = \sqrt[n]{|Q(n)|}$).

Then $\ln |P(n)|^{1/n} = \frac{\ln P(n)}{n}$, and we have:

$$\lim_{n\to\infty} \ln |P(n)|^{1/n} = \lim_{n\to\infty} \frac{\ln P(n)}{n} \overset{\text{L'Hop}}{=} \lim_{n\to\infty} \frac{P'(n)}{P(n)} = 0$$

as $P'(n)$ is a polynomial one degree lower than $P(n)$.

So: $\lim_{n\to\infty} |P(n)|^{1/n} = \lim_{n\to\infty} e^{\ln |P(n)|^{1/n}} = e^0 = 1$.

④ a) $\limsup_{n\to\infty} \left|\frac{2^n}{n^3+1}\right|^{1/n} = \lim_{n\to\infty} \frac{2}{\sqrt[n]{n^3+1}} = \frac{2}{1} = 2$. So $R = \frac{1}{2}$.

b) $\limsup_{n\to\infty} \left|\frac{2n^2+n-1}{3n+4}\right|^{1/n} = \lim_{n\to\infty} \frac{\sqrt[n]{2n^2+n-1}}{\sqrt[n]{3n+4}} = \frac{1}{1} = 1$ So $R = 1$.

c) $\sum_{n=0}^{\infty} n x^{2n} = \sum_{k=0}^{\infty} a_k x^k$ where $a_k = \begin{cases} \frac{k}{2} & \text{if } k \text{ even} \\ 0 & \text{if } k \text{ odd} \end{cases}$

We have $\sup_{k\geq n} \{a_k^{1/k}\} = \sup_{\substack{k\geq n \\ k \text{ even}}} \left\{\left(\frac{k}{2}\right)^{1/k}\right\}$ as $\frac{k}{2} > 0$ for all even k.

And $\sup_{\substack{k\geq n \\ k \text{ even}}} \left\{\left(\frac{k}{2}\right)^{1/k}\right\} = \sup_{2m > n} \left\{m^{\frac{1}{2m}}\right\}$ So $\limsup_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \sqrt[2m]{m} = \sqrt{1} = 1$

So $R = 1$.