4.41

5) a) We have $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ when $|z|<1$.

Choose $z = x^2$. Then $|z|<1 \Leftrightarrow |x|<1$,

so $\sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} x^{2n} = \frac{1}{1-x^2}$ when $|x|<1$.

b) Differentiate the power series from a).

$$\left(\sum_{n=0}^{\infty} x^{2n}\right)' = \sum_{n=0}^{\infty} (x^{2n})' = \sum_{n=0}^{\infty} 2n x^{2n-1}$$

$$\left(\frac{1}{1-x^2}\right)' = -\frac{-2x}{(1-x^2)^2} = \frac{2x}{(1-x^2)^2}$$

So $\frac{2x}{(1-x^2)^2} = \sum_{n=0}^{\infty} 2n x^{2n-1}$ when $|x|<1$.

c) Integrate the power series from a).

$$\int_0^x \sum_{n=0}^{\infty} t^{2n}\, dt = \sum_{n=0}^{\infty} \int_0^x t^{2n}\, dt = \sum_{n=0}^{\infty} \left[\frac{t^{2n+1}}{2n+1}\right]_0^x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$$

$$\frac{1}{1-t^2} = \frac{1}{(1+t)(1-t)} = \frac{A}{1+t} + \frac{B}{1-t} = \frac{A(1-t)+B(1+t)}{(1+t)(1-t)} = \frac{A+B+(B-A)t}{(1+t)(1-t)}$$

So $A+B=1$, $B-A=0$. $\Rightarrow A=\frac{1}{2}, B=\frac{1}{2}$.

$$\int_0^x \frac{1}{1-t^2}\, dt = \int_0^x \frac{\frac{1}{2}}{1+t} + \frac{\frac{1}{2}}{1-t}\, dt = \frac{1}{2}\int_0^x \frac{1}{1+t}\, dt + \frac{1}{2}\int_0^x \frac{1}{1-t}\, dt$$

$$= \frac{1}{2}\Big[\ln|1+t|\Big]_0^x + \frac{1}{2}\Big[-\ln|1-t|\Big]_0^x$$

$$= \frac{1}{2}\left(\ln|1+x| - \ln|1-x|\right) = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|$$

PW

⑥

a) We have, by definition, $\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|c_n|}$;

Let k be any integer.

Then the radius of convergence of $\sum_{n=0}^{\infty} c_n (x-a)^n$ is the same as for $(x-a)^k \sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=0}^{\infty} c_n (x-a)^{n+k}$, as multiplying by a non-zero number does not change whether the series converges.

And $\sum_{n=0}^{\infty} c_n (x-a)^{n+k}$ have a radius of convergence given by

$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n+k]{|c_n|}$, as we wanted.

Note: if $k<0$, this is no longer a power series, but we can just remove the first k terms to turn it into one, without changing the radius of convergence.

b) Have: $\sqrt[n+1]{n+1} = (n+1)^{\frac{1}{n+1}} = e^{\ln(n+1)^{1/n+1}} = e^{\frac{\ln(n+1)}{n+1}}$

By L'Hopital, $\lim_{n\to\infty} \frac{\ln(n+1)}{n+1} = \lim_{n\to\infty} \frac{1}{n+1} = 0$.

So $\lim_{n\to\infty} \sqrt[n+1]{n+1} = \lim_{n\to\infty} e^{\frac{\ln(n+1)}{n+1}} = e^0 = 1$.

c) We have $\limsup_{n\to\infty} \sqrt[n+1]{(n+1)|c_{n+1}|} = \limsup_{n\to\infty} \sqrt[n+1]{(n+1)} \limsup_{n\to\infty} \sqrt[n+1]{c_{n+1}}$

$= 1 \cdot \limsup_{n\to\infty} \sqrt[n]{c_n} = \frac{1}{R}$

$\square$

⑦ a) We have $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ for $|z|<1$, as this is a geometric series.

Choose $z=-x$. Then $\frac{1}{1-z} = \frac{1}{1-(-x)} = \frac{1}{1+x}$ and

$$\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n, \quad \text{when } |-x| = |x| < 1.$$

b) We integrate the series in a), and get:

$$\int_0^x \frac{1}{1+t}\, dt = \Big[\ln|1+t|\Big]_0^x = \ln|1+x|$$

$$\int_0^x \sum_{n=0}^{\infty} (-1)^n t^n\, dt = \sum_{n=0}^{\infty} \int_0^x (-1)^n t^n\, dt = \sum_{n=0}^{\infty} (-1)^n \left[\frac{t^{n+1}}{n+1}\right]_0^x$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

So $\ln|1+x| = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$ when $|x|<1$.

c) We try to put in $x=1$ in the series from b), and get

$\sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1}$. This series converges by the alternating series test.

So by Abels theorem, $\sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} = \lim_{x\to 1} \ln|1+x| = \ln 2$.

4.51

(1)

Want to find $\sup_{x\in[0,1]}\{|x-x^2|\}$.

Differentiate $x-x^2$ to get $1-2x$.

$1-2x=0 \Rightarrow x=\frac{1}{2}$. So $x=\frac{1}{2}$ is a maximum or minimum.

As $(1-2x)'=-2<0$, it is a maximum, and as

$f(0)-g(0)=0$, $f(1)-g(1)=0$, $f(x)-g(x)\geq 0$ on $[0,1]$,

So this is the maximum, and $\rho(f,g)=\frac{1}{2}-\left(\frac{1}{2}\right)^2=\frac{1}{4}$.

(2) Want to find $\sup_{x\in[0,2\pi]}\{|\sin x-\cos x|\}$

Differentiate $\sin x-\cos x$ to get $\cos x+\sin x$.

$\cos x+\sin x=0 \Rightarrow \sin x=-\cos x \Rightarrow \tan x=-1 \Rightarrow x=\frac{3\pi}{4} \vee x=\frac{7\pi}{4}$

Calculate all possible maxima:

$x=0$: $|\sin x-\cos x|=|-1|=1$.

$x=\frac{3\pi}{4}$: $|\sin x-\cos x|=\left|\frac{\sqrt{2}}{2}-\left(-\frac{\sqrt{2}}{2}\right)\right|=\sqrt{2}$

$x=\frac{7\pi}{4}$: $|\sin x-\cos x|=\left|-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\right|=|-\sqrt{2}|=\sqrt{2}$

$x=2\pi$: $|\sin x-\cos x|=|-1|=1$.

So $\rho(f,g)=\sqrt{2}$.

③ Will "cheat" by using the alternative definition of a bounded set.
The set A is bounded if for any c there exists a M_c s.t.
$$d(c,x) \leq M_c \text{ for all } x \in A.$$

So for any $a \in X$, let $c = f(a)$. Then there exists a M_c
s.t. $d_Y(f(a), f(x)) \leq M_c$, as $\{f(x) : x \in X\}$ is bounded.
Choose $M_a = M_c$, and we have shown $\{f(x) : x \in X\}$ bounded $\Rightarrow$ Existence of M_a.

Now assume that for any $a \in X$, there exists a M_a s.t.
$$d_Y(f(a), f(x)) \leq M_a \text{ for any } x \in X.$$
Then, for any $z, w \in \{f(x) | x \in X\}$, there exists $x, y \in X$ s.t.
$f(x) = z$, $f(y) = w$, and $d_Y(f(a), f(x)) \leq M_a$, $d_Y(f(a), f(y)) \leq M_a$.
So $d_Y(z,w) = d_Y(f(x), f(y)) \leq d_Y(f(a), f(x)) + d_Y(f(a), f(y)) \leq 2M_a$.
Therefore, $d_Y(z,w) \leq 2M_a$ for any $z, w \in \{f(x) : x \in X\}$,
which proves that $\{f(x) : x \in X\}$ is bounded.

(5) Choose an $a \in X$. As f_n is bounded, there exists a M_a s.t. $d_Y(f_n(a), f_n(x)) \le M_a$ for all $x \in X$.

As $\rho(f_n, f) < \varepsilon$, $d_Y(f_n(a), f(a)) < \varepsilon$, and $d_Y(f_n(x), f(x)) < \varepsilon$ for all $x \in X$.

So $d_Y(f(a), f(x)) \le d_Y(f(a), f_n(a)) + d_Y(f_n(a), f_n(x)) + d_Y(f_n(x), f(x))$

$$< \varepsilon + M_a + \varepsilon = M_a + 2\varepsilon$$

So choose $M_a' = M_a + 2\varepsilon$, and you have for each $a \in X$ a M_a' s.t. $d_Y(f(a), f(x)) \le M_a'$, i.e. f is bounded.

(6) We have $|x_n| \le M$ and $|y_n| \le N$, so $|x_n - y_n| \le |x_n| + |y_n| \le M + N$, so $\rho(\{x_n\}, \{y_n\}) < \infty$.

Check that ρ is a metric:

(1) Positivity: $\rho(\{x_n\}, \{y_n\}) = \sup_{n \in \mathbb{N}} \{|x_n - y_n|\} \ge 0$

as $|x_n - y_n| \ge 0$.

If $\rho(\{x_n\}\{y_n\}) = 0$, $|x_n - y_n| = 0$ for all n, so $\{x_n\} = \{y_n\}$.

(2) Symmetry: $|x_n - y_n| = |-1||y_n - x_n| = |y_n - x_n|$

So $\rho(\{x_n\}, \{y_n\}) = \sup_{n \in \mathbb{N}} \{|x_n - y_n|\} = \sup_{n \in \mathbb{N}} \{|y_n - x_n|\} = \rho(\{y_n\}, \{x_n\})$.

(3) Triangle inequality:

Let $\{x_n\}, \{y_n\}, \{z_n\}$ be sequences in c_0.

Then:

$$\rho(\{x_n\}, \{z_n\}) = \sup_{n \in \mathbb{N}} \{|x_n - z_n|\} = \sup_{n \in \mathbb{N}} \{|x_n - y_n + y_n - z_n|\}$$

$$\le \sup_{n \in \mathbb{N}} \{|x_n - y_n| + |y_n - z_n|\}$$

$$\le \sup_{n \in \mathbb{N}} \{|x_n - y_n|\} + \sup_{n \in \mathbb{N}} \{|y_n - z_n|\}$$

$$= \rho(\{x_n\}, \{y_n\}) + \rho(\{y_n\}, \{z_n\}),$$ as wanted.

⑥ cont. Want to show that (C_0, ρ) is complete.

Let $\{z_k\}$ be a Cauchy-sequence in C_0, i.e. $z_k = \{x_n^k\}$, where $|x_n^k| \leq M_k$ for all $n \in \mathbb{N}$.

$\{z_k\}$ Cauchy $\Rightarrow$ Given an $\varepsilon > 0$, can find a $K \in \mathbb{N}$ s.t.

$\rho(z_k, z_l) < \varepsilon$ for all $k, l \geq K$,

$\Rightarrow \sup_{n \in \mathbb{N}} \{|x_n^k - x_n^l|\} < \varepsilon$ for all $k, l \geq K$.

$\Rightarrow |x_n^k - x_n^l| < \varepsilon$ for all $k, l \geq K$, for all $n \in \mathbb{N}$.

$\Rightarrow$ For each $n \in \mathbb{N}$, $\{x_n^k\}_{k \in \mathbb{N}}$ is a Cauchy-sequence in $\mathbb{R}$, and will therefore converge to a real number x_n.

Let $z = \{x_n\}$. This is what it looks like:

$$
\begin{array}{rl}
z_1 = & \{x_1^1, x_2^1, x_3^1, \dots, x_n^1, \dots\} \\
z_2 = & \{x_1^2, x_2^2, x_3^2, \dots, x_n^2, \dots\} \\
z_3 = & \{x_1^3, x_2^3, x_3^3, \dots, x_n^3, \dots\} \\
\vdots & \\
z_k = & \{x_1^k, x_2^k, x_3^k, \dots, x_n^k, \dots\} \\
 & \quad \downarrow \quad \downarrow \quad \downarrow \qquad\quad \downarrow \\
z = & \{x_1, x_2, x_3, \dots, x_n, \dots\}
\end{array}
$$

This seems like a likely candidate for $\lim_{k \to \infty} z_k$

<u>Must show:</u> (I) $z \in C_0$, i.e. z is bounded.

(II) $\lim_{k \to \infty} z_k = z$, i.e. z is indeed the correct choice.

If both hold, we have that z_k converges so C_0 is complete.

⑥ cont.

(I) Have: $|x_n^k| \le M_k$ for all $n \in \mathbb{N}$, as z_k bounded.

$|x_n^k - x_n^l| < \varepsilon$ for all $k, l \ge K$, as $\{z_k\}$ Cauchy.

So: $|x_n^k - x_n^K| < \varepsilon$ (choose $l = K$), $k \ge K$, $n \in \mathbb{N}$

$|x_n^k| < |x_n^K| + \varepsilon$ (Triangle inequality on $|x_n^k - x_n^K + x_n^K|$), $k \ge K$

$|x_n^k| < M_K + \varepsilon$ ($\{x_n^K\} = z_K$ bounded) $k \ge K$, $n \in \mathbb{N}$.

Then: $|x_n^k| \le M = \max\limits_{l \le K}\{M_l + \varepsilon\}$ for all $n \in \mathbb{N}, k \in \mathbb{N}$,

as $k \ge K \Rightarrow |x_n^k| \le M_K + \varepsilon \le \max\limits_{l \le K}\{M_l + \varepsilon\}$

and $k \le K \Rightarrow |x_n^k| \le M_k \le M_k + \varepsilon \le \max\limits_{l \le K}\{M_l + \varepsilon\}$.

Now, for any $n \in \mathbb{N}$, $\lim\limits_{k \to \infty} x_n^k = x_n$, so can find K_n s.t.

$|x_n - x_n^k| < \varepsilon$ when $k \ge K_n$

Then $|x_n| \le |x_n^k| + \varepsilon \le M + \varepsilon = M'$ for all $n \in \mathbb{N}$,

so z is bounded, $z \in C_0$.

(II) Have: $|x_n^k - x_n^l| < \frac{\varepsilon}{2}$ for $k, l \ge K'$, as $\{z_k\}$ Cauchy.

$|x_n^l - x_n| < \frac{\varepsilon}{2}$ for $l \ge K_n$, as $\lim\limits_{l \to \infty} x_n^l = x_n$

So: $|x_n^k - x_n| \le |x_n^k - x_n^l| + |x_n^l - x_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for $k \ge K'$, for all $n \in \mathbb{N}$.

Then: $\sup\limits_{n \in \mathbb{N}}\{\{x_n^k\}, \{x_n\}\} = \rho(z_k, z) < \varepsilon$ when $k \ge K'$,

so $\lim\limits_{k \to \infty} z_k = z$.

7

a) f uniformly continuous $\iff$ For all $\varepsilon > 0$, there exists a $\delta > 0$ s.t. $|x-y| < \delta \Rightarrow |f(x)-f(y)| < \varepsilon$.

Let $\varepsilon > 0$, choose δ as here $\nearrow$.

Choose any $\{r_n\}$ s.t. $\lim_{n\to\infty} r_n = 0$. Can find an N s.t. $|r_n| < \delta$ when $n \geq N$, so $|f_{r_n}(x) - f(x)| < \varepsilon$ as $|f_{r_n}(x)-f(x)| = |f(x+r_n)-f(x)|$ and $|x+r_n-x| = |r_n| < \delta$.

As this is true for all $x \in \mathbb{R}$, $\rho(f_{r_n}, f) < \varepsilon$, when $n \geq N$, so

$\lim_{n\to\infty} \rho(f_{r_n}, f) = \lim_{r\to 0} \rho(f_r, f) = 0$.

b) Want to show that $\cos(\pi x^2)$ do not satisfy the condition in a).

We know that $\cos(\pi x^2)$ is 1 if x^2 is an even number, and $\cos(\pi x^2) = -1$ if x^2 is an odd number. So if we can find x_n and r_n s.t. $r_n \to 0$ and x_n^2 is even, $(x_n+r_n)^2$ is odd, then $\lim_{r\to 0} \rho(f_r, f) \geq \lim_{n\to\infty} |f(x_n+r_n) - f(x_n)| = 2 \neq 0$.

As a first try, let $x_n = 2n$. We want $(x_n+r_n)^2 = x_n^2 + 1$, so must solve $r_n^2 + 2x_n r_n = 1 \Rightarrow r_n = \sqrt{x_n^2+1} - x_n = \sqrt{4n^2+1} - 2n$

Then $r_n = \dfrac{(\sqrt{x_n^2+1} - x_n)(\sqrt{x_n^2+1} + x_n)}{\sqrt{x_n^2+1} + x_n} = \dfrac{x_n^2+1-x_n^2}{\sqrt{x_n^2+1}+x_n} \leq \dfrac{1}{2x_n} = \dfrac{1}{4n}$,

so $r_n \to 0$ as $n \to \infty$.

So for any $\delta > 0$, there exists $r_n < \delta$, $x_n \in \mathbb{R}$ s.t. $|f_{r_n}(x_n) - f(x_n)| = 2$, and $\rho(f_r(x), f(x))$ cannot go to 0 as $r \to 0$, and $\cos(\pi x^2)$ cannot be uniformly convergent.

c) No, $\cos(\pi x^2)$ is a counterexample.

4.6

① Let $f(x)=x$, $g(x)=0$. Then $|f(x)-g(x)|=|x| \to \infty$ as $x\to\infty$

② We have the following: If $a\in\mathbb{R}^n$, $f_a(x)=d(x,a)$ is continuous.
If X is not compact, then either: X is not closed
or X is not bounded. as $X\subseteq\mathbb{R}^n$

X not closed: There exists an $a\in\mathbb{R}^n$, $a\notin X$
and a sequence $\{x_n\}\subseteq X$ with $x_n\to a$.
Let $f(x)=\frac{1}{f_a(x)}=\frac{1}{d(x,a)}$. As $d(x,a)\neq 0$ on X,
this is continuous. And as $d(x_n,a)\to 0$, $f(x_n)\to\infty$, and
f is not bounded.

X not bounded: We can find a sequence $\{x_n\}\subseteq X$
s.t. $d(x_1,x_n)\to\infty$ as $n\to\infty$.
Choose $f(x)=d(x_1,x)$. This is continuous and $f(x_n)\to\infty$.

(3)

a) Want to show that $L(u)$ is a continuous function from $[0,1]$ to $\mathbb{R}$. As you can give it values t from $[0,1]$ and get values in $\mathbb{R}$, all we need to do is show that it's continuous.

Let $\varepsilon > 0$. Want $|L(u)(t) - L(u)(t')| < \varepsilon$ when $|t-t'| < \delta$.

Have $f(x)$ is bounded, so $|f(x)| \leq M$.

So
$$|L(u)(t) - L(u)(t')| = \left| \int_0^1 \left(\frac{1}{1+t+s} - \frac{1}{1+t'+s} \right) f(u(s))\, ds \right|$$
$$\leq \int_0^1 \left| \frac{1}{1+t+s} - \frac{1}{1+t'+s} \right| M\, ds$$
$$= \int_0^1 \frac{|t'-t|}{|(1+t+s)(1+t'+s)|} M\, ds$$
$$\leq M|t'-t| \int_0^1 \frac{1}{(1+s)^2}\, ds$$
$$= M|t'-t| \cdot \frac{1}{2} < \frac{M\delta}{2} = \varepsilon \quad \text{if we choose } \delta = \frac{2\varepsilon}{M}.$$

So $L(u)$ is continuous.

b) Want to show that L is a contraction, so we can use Banachs fixed point thm.

Have:
$$|L(u)(t) - L(v)(t)| = \left| \int_0^1 \frac{1}{1+t+s} \left(f(u(s)) - f(v(s)) \right) ds \right|$$
$$\leq \int_0^1 \frac{1}{1+t+s} |f(u(s)) - f(v(s))|\, ds$$
$$\leq \int_0^1 \frac{1}{1+t+s} \frac{c}{\ln 2} |u(s) - v(s)|\, ds$$
$$\leq \frac{c}{\ln 2} \sup_{s \in [0,1]} |u(s) - v(s)| \int_0^1 \frac{1}{1+t+s}\, ds$$
$$= \frac{c}{\ln 2} \rho(u,v) \ln \left| \frac{2+t}{1+t} \right|$$
$$\leq C \rho(u,v) \quad \text{as } t \geq 0.$$

③ b) cont: So $|L(u)(t) - L(v)(t)| \le C\rho(u,v)$ for all $t \in [0,1]$

$\Rightarrow \rho(L(u), L(v)) \le C \cdot \rho(u,v)$

$\Rightarrow L$ is a contraction, $C([0,1], \mathbb{R})$ is complete,

so $L(u) = u$ has a unique solution.

④

a) Ⅰ Non-negativity:

As $d(x,y) \ge 0$, $\bar{d}(x,y) = \min\{d(x,y), 1\} \ge 0$,

and $\bar{d}(x,y) = 0 \Leftrightarrow d(x,y) = 0 \Leftrightarrow x = y$.

Ⅱ Symmetry:

If $d(x,y) \le 1$, $\bar{d}(x,y) = d(x,y) = d(y,x) = \bar{d}(y,x)$

If $d(x,y) > 1$, $d(y,x) > 1$, so $\bar{d}(x,y) = 1 = \bar{d}(y,x)$.

Ⅲ Triangle inequality:

If $d(x,z) \le 1$, $\bar{d}(x,z) = d(x,z) \le d(x,y) + d(y,z)$

If both $d(x,y), d(y,z) \le 1$, we have $\bar{d}(x,z) \le \bar{d}(x,y) + \bar{d}(y,z)$.

If either $d(x,y), d(y,z) > 1$, $\bar{d}(x,y) + \bar{d}(y,z) \ge 1 \ge \bar{d}(x,z)$.

If $d(x,z) > 1$, $1 \le d(x,y) + d(y,z)$.

If both $d(x,y), d(y,z) \le 1$, we have $\bar{d}(x,z) = 1 \le \bar{d}(x,y) + \bar{d}(y,z)$

If either $d(x,y), d(y,z) > 1$, $\bar{d}(x,y) + \bar{d}(y,z) \ge 1 = \bar{d}(x,z)$.

b) Let G be open in d-metric. Let $x \in G$. Want to show that x is an inner point in the $\bar{d}$-metric.

As G is open, there exists an $\varepsilon > 0$ s.t. $B_d(x,\varepsilon) \subseteq G$.

Let $\varepsilon' = \min\{\varepsilon, 0.5\}$. Then, if $y \in B_{\bar{d}}(x,\varepsilon')$, $\bar{d}(x,y) = d(x,y) < \varepsilon' \le \varepsilon$,

so $y \in B_d(x,\varepsilon)$. Therefore, $B_{\bar{d}}(x,\varepsilon') \subseteq B_d(x,\varepsilon) \subseteq G$, and x is an inner point.

If G is open in the $\bar{d}$-metric, let $x \in G$, $\varepsilon > 0$ s.t. $B_{\bar{d}}(x,\varepsilon) \subseteq G$.

If $\varepsilon \ge 1$, $B_{\bar{d}}(x,\varepsilon) = Y$, so $B_d(x,\varepsilon) \subseteq B_{\bar{d}}(x,\varepsilon)$.

If $\varepsilon < 1$, $d(x,y) = \bar{d}(x,y)$ for all $y \in B_{\bar{d}}(x,\varepsilon)$, so $B_d(x,\varepsilon) = B_{\bar{d}}(x,\varepsilon)$.

In either case, x is an inner point.

Closed sets are just complements of open sets, so if the open sets are the same, closed sets must be the same as well.

④ cont.

c) Have: $\{z_n\}$ converges to a in d-metric $\Leftrightarrow \lim_{n\to\infty} d(z_n,a) = 0$

If $\lim_{n\to\infty} d(z_n,a) \to 0$, we must have an $N \in \mathbb{N}$ s.t. $d(z_n,a) < 1$ for $n \geq N$.

So $d(z_n,a) = \bar{d}(z_n,a)$ for $n \geq N$, and $\lim_{n\to\infty} \bar{d}(z_n,a) = \lim_{n\to\infty} d(z_n,a) = 0$,

so $\{z_n\}$ converges to a in $\bar{d}$-metric.

The same is true the other way as well.

d) $\{y_n\}$ Cauchy in $(Y,d) \Leftrightarrow$ For all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ s.t. $d(y_n,y_m) < \varepsilon$ when $n,m \geq N$.

If $\varepsilon > 1$, $\bar{d}(y_n,y_m) < d(y_n,y_m)$, and if $\varepsilon \leq 1$, $\bar{d}(y_n,y_m) = d(y_n,y_m)$.

Either way, $\bar{d}(y_n,y_m) < \varepsilon$ when $n,m \geq N$, so $\{y_n\}$ is Cauchy in $(Y,\bar{d})$.

$\{y_n\}$ Cauchy in $(Y,\bar{d}) \Leftrightarrow$ For all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ s.t. $\bar{d}(y_n,y_m) < \varepsilon$ when $n,m \geq N$.

Choose $\varepsilon' = \min(\varepsilon, 0.5)$, find $N' \in \mathbb{N}$ s.t. $\bar{d}(y_n,y_m) < \varepsilon'$ when $n,m \geq N'$.

Then $d(y_n,y_m) = \bar{d}(y_n,y_m) < \varepsilon' \leq \varepsilon$ when $n,m \geq N'$, so $\{y_n\}$ is Cauchy in (Y,d).

So $\{y_n\}$ Cauchy in $\bar{d} \Leftrightarrow \{y_n\}$ Cauchy in d.

If (Y,d) is complete, $\{y_n\}$ converges in $d \Leftrightarrow \{y_n\}$ converges in $\bar{d}$.

If $(Y,\bar{d})$ is complete, $\{y_n\}$ converges in $\bar{d} \Leftrightarrow \{y_n\}$ converges in d.

(4) cont.

e) If K compact in (Y,d), and $\{x_n\}$ is a sequence in K, $\{x_n\}$ has a convergent subsequence in d. By c), this converges in $\bar{d}$ as well, so $\{x_n\}$ has a convergent subsequence in $\bar{d}$. This is true for all $\{x_n\} \subseteq K$, so K is compact in $\bar{d}$. The other implication is identical. (K compact in $\bar{d} \Rightarrow K$ compact in d)

f) If f continuous at a point a in d, we have:

$\{x_n\}$ converges to a in $\bar{d} \Leftrightarrow \{x_n\}$ converges to a in d

$\Rightarrow \{f(x_n)\}$ converges to $f(a)$ in $d \Leftrightarrow \{f(x_n)\}$ converges to $f(a)$ in $\bar{d}$,

So f continuous at a in $\bar{d}$. The other implication (f cont. in $\bar{d} \Rightarrow f$ cont in d) is identical.

If g continuous in d, we have:

If A open in X, $g^{-1}(A)$ open in $(Y,d) \Rightarrow g^{-1}(A)$ open in $(Y,\bar{d})$

So g continuous in $\bar{d}$. The other implication is again identical.

g) Let us distinguish between $C(X, Y_d)$ where we give Y the d-metric and $C(X, Y_{\bar{d}})$, where we use the $\bar{d}$-metric.

By f), $f \in C(X, Y_d) \Leftrightarrow f \in C(X, Y_{\bar{d}})$. The big difference between these is that in $C(X, Y_{\bar{d}})$, all functions are bounded, so $C(X, Y_{\bar{d}}) = C_b(X, Y_{\bar{d}})$.

We have $\bar{\rho}_d(f,g) = \rho_{\bar{d}}(f,g)$, so we have that

$$(C(X, Y_d), \bar{\rho}) = (C_b(X, Y_{\bar{d}}), \rho),$$

and this last space is complete by Thm 4.6.2, as (Y,d) complete implies $(Y,\bar{d})$ complete.