4.7

① $y' = 1 + y^2 \Rightarrow \dfrac{y'}{1+y^2} = 1$

$$\Rightarrow \int \frac{y'}{1+y^2}\,dx = \int dx \Rightarrow \int \frac{1}{1+y^2}\,dy = \int dx$$

$$\Rightarrow \arctan y = x + C$$

$$y = \tan(x+C)$$

$$y(0) = \tan(C) = 0 \Rightarrow C = 0$$

$\tan x$ is undefined at $x = \frac{\pi}{2}$, and as $\lim\limits_{x \to \frac{\pi}{2}} \tan x = \infty$ we cannot find a fitting value for y at $\frac{\pi}{2}$ s.t. y is differentiable.

② $0' = 0 = \frac{3}{2} 0^{1/3}$, so the equation is satisfied for $t < a$.

$\left((t-a)^{\frac{3}{2}}\right)' = \frac{3}{2}(t-a)^{1/2} = \frac{3}{2}\left((t-a)^{3/2}\right)^{1/3}$, so the equation is satisfied for $t > a$.

$$y'(a) = \lim_{h \to 0} \frac{y(a+h) - y(a)}{h} = \frac{y(a+h)}{h}$$

From below:

$$\lim_{h \nearrow 0} \frac{y(a+h)}{h} = \frac{0}{h} = 0$$

From above:

$$\lim_{h \searrow 0} \frac{y(a+h)}{h} = \lim_{h \searrow 0} \frac{(a+h-a)^{3/2}}{h} = \lim_{h \searrow 0} h^{1/2} = 0$$

So $y'(a) = 0 = \frac{3}{2}(y(a))^{1/3}$

③

If we have $y'(t) = f(t, y(t))$, then

$$\begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} v(t) \\ g(t, u(t), v(t)) \end{bmatrix} \Rightarrow \begin{array}{l} u'(t) = v(t) \\ v'(t) = u''(t) \\ u''(t) = g(t, u(t), u''(t)) \end{array}$$

and $y(0) = \begin{bmatrix} u(0) \\ v(0 \end{bmatrix} = \begin{bmatrix} u(0) \\ u'(0) \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow \begin{array}{l} u(0) = a \\ u'(0) = b \end{array}$

So u is a solution to ⊛.

4.8|

① We know that $\mathbb{Q}$ is countable and dense in $\mathbb{R}$, so let's hope $\mathbb{Q}^n$ is countable and dense in $\mathbb{R}^n$

A countable cartesian product of a countable set is countable, so $\mathbb{Q}^n$ is countable.

If $x=(x_1,\dots,x_n)\in\mathbb{R}^n$, want to find $q=(q_1,\dots,q_n)\in\mathbb{Q}^n$ s.t. $\|x-q\|<\varepsilon$.

As $\mathbb{Q}$ is dense in $\mathbb{R}$, can find $q_i\in\mathbb{Q}$ s.t. $|x_i-q_i|<\frac{\varepsilon}{n}$ for all $i=1,\dots,n$. Let $q=(q_1,\dots,q_n)$. Then:

$$\|x-q\|=\sqrt{(x_1-q_1)^2+\dots+(x_n-q_n)^2}<\sqrt{\frac{\varepsilon^2}{n^2}+\dots+\frac{\varepsilon^2}{n^2}}=\sqrt{\frac{\varepsilon^2}{n}}=\frac{\varepsilon}{\sqrt{n}}<\varepsilon,$$

so $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$ and $\mathbb{R}^n$ is separable.

② A is dense in X if for each $x\in X$ there is an $\{a_n\}\subseteq A$ s.t. $\lim_{n\to\infty} a_n=x$.

Then, for $a\in X$, there is an $\{a_n\}\subseteq A$ s.t. $\lim_{n\to\infty} a_n=a$, so given an $r>0$, can find an $N\in\mathbb{N}$ s.t. $d(a_n,a)<r$ when $n\geq N$, i.e. $B(a,r)$ contains $a_N\in A$. This is true for all $a\in X$, $r>0$.

If all open balls $B(a,r)$, $a\in X$, $r>0$ contain elements from A, we can, for $a\in X$ find $a_n\in B(a,\frac{1}{n})$, $a_n\in A$.

And then $\lim_{n\to\infty} d(a_n,a)\leq\lim_{n\to\infty}\frac{1}{n}=0$, so $a_n\to a$.

This works for all $a\in X$, so A is dense.

③

a) To show that $\{f(a_n)\}$ converges, it's enough to check that it's Cauchy, as $\mathbb{R}$ is complete. Let $\varepsilon > 0$.

As f is uniformly continuous, we have that

$$|f(a_n) - f(a_m)| < \varepsilon \text{ whenever } d(a_n, a_m) < \delta.$$

As $\{a_n\}$ converges it's Caucy, so we can find N s.t.

$$d(a_n, a_m) < \delta \text{ wen } n, m \geq N.$$

And then $|f(a_n) - f(a_m)| < \varepsilon$ whenever $n, m \geq N$, so $\{f(a_n)\}$ is Cauchy, and must therefore converge.

If $\{a_n\}$ converges to $x \in X$, and $\{b_n\}$ converges to x, must check that $\{f(a_n)\}$ and $\{f(b_n)\}$ converge to the same element. If $\lim_{n \to \infty} |f(a_n) - f(b_n)| = 0$ this must be the case.

Let $\varepsilon > 0$. Want to find N s.t. $|f(a_n) - f(b_n)| < \varepsilon$ whenever $n \geq N$. As f is uniformly continuous, have that $|f(a_n) - f(b_n)| < \varepsilon$ whenever $d(a_n, b_n) < \delta$. As $\{a_n\}$ converges to x, can find an N_1 s.t. $d(a_n, x) < \frac{\delta}{2}$ when $n \geq N_1$, and as $\{b_n\}$ converges to x, can find an N_2 s.t. $d(x, b_n) < \frac{\delta}{2}$ when $n \geq N_2$. Let $N = \max\{N_1, N_2\}$.

Then $d(a_n, b_n) \leq d(a_n, x) + d(x, b_n) < \frac{\delta}{2} + \frac{\delta}{2} = \delta$ whenever $n \geq N$, so $|f(a_n) - f(b_n)| < \varepsilon$ whenever $n \geq N$, and $f(a_n)$ must converge to the same thing as $f(b_n)$.

③ cont.

b) For $a,b \in A$, we have $\bar{f}(a) = f(a)$, $\bar{f}(b) = f(b)$, so we have $|f(a) - f(b)| < \varepsilon/3$ whenever $d(a,b) < \delta$, $a,b \in A$. Want to show that there exists a $\delta' > 0$ s.t. $|\bar{f}(x) - \bar{f}(y)| < \varepsilon$ whenever $d(x,y) < \delta'$.

Choose an $\{a_n\}$ converging to x and a $\{b_n\}$ converging to y.
Then $f(a_n)$ converges to $\bar{f}(x)$ and $f(b_n)$ converges to $\bar{f}(b)$.
So we can find an N s.t. $d(a_n, x) < \frac{\delta}{3}$, $d(y, b_n) < \frac{\delta}{3}$, $|\bar{f}(x) - f(a_n)| < \frac{\varepsilon}{3}$ and $|f(b_n) - \bar{f}(y)| < \frac{\varepsilon}{3}$. If we now choose $\delta' = \frac{\delta}{3}$ we have

$$d(a_N, b_N) \leq d(a_N, x) + d(x,y) + d(y, b_N) < \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta,$$ so

we have $|f(a_N) - f(b_N)| < \frac{\varepsilon}{3}$. And then we have

$$|\bar{f}(x) - \bar{f}(y)| \leq |\bar{f}(x) - f(a_N)| + |f(a_N) - f(b_N)| + |f(b_N) - \bar{f}(y)|$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$ as we wanted.

This works for all $x, y \in X$, so we have that $\bar{f}$ is uniformly convergat.

c) Want to show that for any $q \in \mathbb{Q}$ we can find a $\delta > 0$ s.t. $|f(a) - f(q)| < \varepsilon$ whenever $|a - q| < \delta$.

Given $q \in \mathbb{Q}$, we will choose $\delta = |q - \sqrt{2}|$. Then, if $|a - q| < \delta$, either both a and q are $< \sqrt{2}$ or both are $> \sqrt{2}$. In both cases $f(a) = f(q)$, so $|f(a) - f(q)| = 0 < \varepsilon$.

It cannot be uniformly continuous, as we can find two different sequences of rational numbers, $\{a_n\}$ and $\{b_n\}$, one converging to $\sqrt{2}$ from below and the other from above. But $f(a_n) \to 0$ and $f(b_n) \to 1$, and $0 \neq 1$. This could not happen if f was uniformly continuous by a).

d) For continuous functions, $f(a_n) \to f(a)$ when $a_n \to a$, but as mentioned we can find two different sequences both converging to $\sqrt{2}$, with $f(a_n) \to 0$, $f(b_n) \to 1$. So we cannot continuously assign a value to $\sqrt{2}$!

④ That any sequence $\{f_n\}$ of contractions have a uniformly convergent subsequence, must mean that the set of all contractions as a subset of $C(K,K)$, must be compact if you take the closure.

<u>Note:</u> The set of all contractions are <u>not</u> closed.

<u>Example:</u> If $K=[0,1]\subset\mathbb{R}$, $f_n(x)=(1-\frac{1}{n})x$, then $\{f_n\}$ is a sequence of contractions, converging to $f(x)=x$, and f is <u>not</u> a contraction.

Therefore, we let $\mathcal{K}$ be the closure of the set of contractions in $C(K,K)$. Want to show that $\mathcal{K}$ is compact, and will therefore use Arzela-Ascoli's thm. Must show that $\mathcal{K}$ is closed, bounded and eqicontinuous.

<u>Closed:</u> $\mathcal{K}$ is defined as the closure of the set of contractions, and closures of sets are always closed.

<u>Bounded:</u> As K is compact it's bounded, so there exists an M s.t. $\|x-y\|\leq M$ for all $x,y\in K$. As any two functions in $\mathcal{K}$ are functions from K to K, $\rho(f,g)=\sup\limits_{x\in K}\{\|f(x)-g(x)\|\}\leq\sup\limits_{x,y\in K}\{\|x-y\|\}\leq M$. This is true for all $f,g\in\mathcal{K}$, so $\mathcal{K}$ is bounded.

<u>Eqicontinuous:</u> If $f\in\mathcal{K}$, either f is a contraction and $\|f(x)-f(y)\|\leq s\|x-y\|$ for $s<1$, or $f=\lim\limits_{n\to\infty} f_n$ where f_n are contractions and $\|f_n(x)-f_n(y)\|\leq s_n\|x-y\|$ with $s_n<1$ for all n. And then

$$\|f(x)-f(y)\|=\lim_{n\to\infty}\|f_n(x)-f_n(y)\|\leq\sup_{n\in\mathbb{N}}\{s_n\}\|x-y\|\leq\|x-y\|$$

So in either case $\|f(x)-f(y)\|\leq\|x-y\|$. So given an $\varepsilon>0$, if we let $\delta=\varepsilon$, $\|f(x)-f(y)\|\leq\|x-y\|<\delta=\varepsilon$ whenever $\|x-y\|<\delta$, for all $f\in\mathcal{K}$, so $\mathcal{K}$ is equicontinuous.

Therefore $\mathcal{K}$ is compact, any sequence $\{f_n\}$ of contractions is a sequence in $\mathcal{K}$ and must have a convergent subsequence, and as the metric is the ρ-metric, the convergence must be uniform.

(b)

Want to show that K is compact, i.e. closed, bounded and equicontinuous.

<u>Closed:</u> Let $\{f_n\}$ be a sequence in K converging to an f in $C([-1,1],\mathbb{R})$. Want to show that $f \in K$.

Have $|f(x)-f(y)| = \lim_{n\to\infty} |f_n(x)-f_n(y)| \leq \lim_{n\to\infty} K|x-y| = K|x-y|$,

so f is Lipschitz-continuous with Lipschitz-constant K.

And $f(0) = \lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} 0 = 0$. So $f \in K$.

<u>Bounded:</u> For any $f, g \in K$, we have:

$$|f(x)-g(x)| \leq |f(x)-f(0)| + |f(0)-g(0)| + |g(0)-g(x)|$$
$$\leq K|x-0| + K|0-x| = 2K|x| \leq 2K.$$

This is true for all $f, g \in K$, so K is bounded.

<u>Equicontinuous:</u> Given an ε, we have $|f(x)-f(y)| \leq K|x-y| < K\delta = \varepsilon$ for $\delta = \frac{\varepsilon}{K}$, when $|x-y|<\delta$.

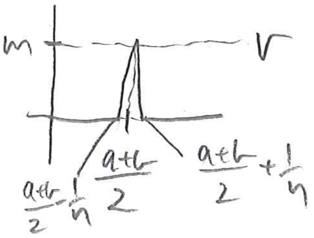
This is true for all $f \in K$, so K is equicontinuous.

As K is closed, bounded and equicontinuous, K is compact, by Arzela-Ascolis theorem.

⑦ In $\mathbb{R}^m$, any closed ball $\overline{B}(a,r)$ is closed and bounded, and therefore compact by Bolzano-Weierstrass.

For $C([a,b],\mathbb{R})$, we need to show that $\overline{B}(f,r)$ is not compact for some f.
$\overline{B}(f,r)$ is closed and bounded, so need to show it's not equicontinuous.

Let us study the functions f_n of the form



Let f be given by $f(x)=0$.
Then $\rho(f,f_n)=r$, so $f_n \in \overline{B}(f,r)$.
So for us to have $|f_n(x)-f_n(y)|<\varepsilon'<r$, for $y=\frac{a+b}{2}$, we must have $\delta<\frac{1}{n}$. But then we cannot use the same δ for all functions in $\overline{B}(f,r)$, as we would need $\delta<\frac{1}{n}$ for all $n\in\mathbb{N}$, which is impossible.
So $\overline{B}(f,r)$ is not compact for any $r>0$.

4.91

①

Let us write $f(x) = (f_1(x), \ldots, f_m(x))$.

Each $f_i(x)$ is a function from a compact into $\mathbb{R}$, and must therefore have a maximum and minimum, so we know that $|f_i(x)| < K_i$ for each i. Then:

$$\|f(x)\| = \sqrt{|f_1(x)|^2 + \cdots + |f_m(x)|^2} \leq \sqrt{K_1^2 + \cdots + K_m^2} = K'.$$

So f is bounded. Now, as f_n converges uniformly to f, given $\varepsilon > 0$ we can choose an N s.t. $\|f(x) - f_n(x)\| < \varepsilon$ for all $n \geq N$ and all $x \in [a,b]$.

So $\|f_n(x)\| \leq \|f_n(x) - f(x)\| + \|f(x)\| < \varepsilon + K'$ when $n \geq N$.

Each f_n for $n \leq N$ must be bounded, by the same logic as for f, so $\|f_n(x)\| \leq M_n$ for each $n \leq N$.

Now let $K = \max(M_1, \ldots, M_N, K' + \varepsilon)$.

Then $\|f_n(x)\| \leq K$ for all $n \in \mathbb{N}$ and all $x \in [a,b]$.

②

In 4.7.1, $f(t,y) = 1 + y^2$ which is continuous.

In 4.7.2, $f(t,y) = \frac{3}{2} y^{1/3}$ which is continuous.

In 4.7.1, the solution is only valid on an interval, and in 4.7.2, the solution is not unique, so neither of these functions can be uniformly Lipschitz.