4.10

① Assume that we have a sequence $p_n(x)$ converging uniformly to $\frac{1}{x}$ on $(0,1)$.
Then $\lim_{n\to\infty} \sup_{x\in(0,1)} |p_n(x) - \frac{1}{x}| = 0$, i.e. given an $\varepsilon > 0$,
we can find an $N \in \mathbb{N}$ s.t. $\sup_{x\in(0,1)} |p_n(x) - \frac{1}{x}| < \varepsilon$ when $n \geq N$.

But we have $\lim_{x\to 0} p_n(x) = p_n(0)$, as polynomials are continuous, and
$\lim_{x\to 0} \frac{1}{x} = \infty$. So $\lim_{x\to 0} |\frac{1}{x} - p_N(x)| = \infty$, which means that we
can find an $x \in (0,\delta)$ s.t. $|\frac{1}{x} - p_N(x)| > \varepsilon$.
And then $\sup_{x\in(0,1)} |p_N(x) - \frac{1}{x}| > \varepsilon$, contradicting our earlier claim.
So there cannot exist a sequence of $p_n(x)$ converging uniformly
to $\frac{1}{x}$ on $(0,1)$.

② Assume that we have a sequence $p_n(x)$ converging uniformly to e^x
on $\mathbb{R}$. Then $\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |p_n(x) - e^x| = 0$, i.e. given an $\varepsilon > 0$,
we can find an $N \in \mathbb{N}$ s.t. $\sup_{x\in\mathbb{R}} |p_n(x) - e^x| < \varepsilon$ when $n \geq N$.
But we have $\lim_{x\to-\infty} |p_N(x)| = \infty$, as all polynomials tend to + or $-\infty$
when x tend to + or $-\infty$, and $\lim_{x\to-\infty} e^x = 0$.
So $\lim_{x\to-\infty} |p_N(x) - e^x| = \infty$, i.e. we can find a $K \in \mathbb{R}$ s.t. $x < -K$
implies $|p_N(x) - e^x| > \varepsilon$. ~~And~~ And then $\sup_{x\in\mathbb{R}} |p_N(x) - e^x| > \varepsilon$,
contradicting our earlier statement. So there cannot
exist a sequence $p_n(x)$ converging uniformly to e^x.

③ a) Will prove this inductively:

$$n=0: \quad f^{(0)}(x) = e^{-1/x^2}\frac{P_0(x)}{x^{N_0}} \quad \text{with } P_0(x)=1, \; N_0 = 0$$

$$= e^{-1/x^2}$$

Assume ok for $n=k$.

$n=k+1$:

$$f^{(k+1)}(x) = \left(f^{(k)}(x)\right)' = \left(e^{-1/x^2}\frac{P_k(x)}{x^{N_k}}\right)'$$

$$= \frac{2}{x^3}e^{-1/x^2}\frac{P_k(x)}{x^{N_k}} + e^{-1/x^2}\left(\frac{P_k'(x)x^{N_k} - N_k x^{N_k-1}P_k(x)}{x^{2N_k}}\right)$$

$$= e^{-1/x^2}\left(\frac{2P_k(x)}{x^{N_k+3}} + \frac{P_k'(x)x^{N_k} - N_k x^{N_k-1}P_k(x)}{x^{2N_k}}\right)$$

$$= e^{-1/x^2}\frac{P_{k+1}(x)}{x^{N_{k+1}}}$$

where $N_{k+1} = \max(N_k+3, 2N_k)$

and $P_{k+1}(x) = 2P_k(x)\,x^{N_k-3} + P_k'(x)x^{N_k} - N_k x^{N_k-1}P_k(x)$

if $2N_k \geq N_k+3 \Leftrightarrow N_k \geq 3$

or $P_{k+1}(x) = 2P_k(x) + \left(P_k'(x)x^{N_k} - N_k x^{N_k-1}P_k(x)\right)x^{3-N_k}$

if $N_k+3 > 2N_k \Leftrightarrow 3 > N_k$;

③ cont.

b) Will again show this inductively.

$n=0$: $f^{(0)}(0) = f(0) = 0$ by definition.

Assume OK for $n=k$:

$n=k+1$:

$$f^{(k+1)}(0) = \left(f^{(k)}\right)'(0) = \lim_{h\to 0} \frac{f^{(k)}(h) - f^{(k)}(0)}{h}$$

$$= \lim_{h\to 0} \frac{e^{-1/h^2} P_k(h)}{h^{N_k+1}} = P_k(0) \lim_{h\to 0} \frac{e^{-1/h^2}}{h^{N_k+1}}$$

$$= P_k(0)\cdot 0 = 0$$

Why is $\lim_{h\to 0} \frac{e^{-1/h^2}}{h^{N_k+1}} = 0$? Will use induction (again) on N_k.

$N_k = 0$: $\lim_{h\to 0} \frac{e^{-1/h^2}}{h^1} = \lim_{h\to 0} \frac{h^{-1}}{e^{1/h^2}} \overset{L'H}{=} \lim_{h\to 0} \frac{-h^{-2}}{-\frac{2}{h^3} e^{1/h^2}}$

$$= \lim_{h\to 0} \frac{h}{2e^{1/h^2}} = \frac{0}{\infty} = 0.$$

Assume $\lim_{h\to 0} \frac{e^{-1/h^2}}{h^{n+1}} = 0$ for all $n \le k$. Show $\lim_{h\to 0} \frac{e^{-1/h^2}}{h^{(k+1)+1}} = 0$

$$\lim_{h\to 0} \frac{e^{-1/h^2}}{h^{k+2}} = \lim_{h\to 0} \frac{h^{-k-2}}{e^{1/h^2}} \overset{L'H}{=} \lim_{h\to 0} \frac{-(k+2)h^{-k-3}}{-\frac{2}{h^3} e^{1/h^2}}$$

$$= \lim_{h\to 0} \frac{(k+2)h^{-k}}{2\, e^{1/h^2}} = \frac{k+2}{2} \lim_{h\to 0} \frac{e^{-1/h^2}}{h^{(k-1)}} = \frac{k+2}{2}\cdot 0 = 0$$

by induction hypothesis.

(3)

c) Taylor polynomial of f at 0:

$$T(f)(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n = \sum_{n=0}^{\infty} \frac{0}{n!} x^n = 0$$

So as $f(x) \neq 0$ for $x \neq 0$, we have that Tf does not converge to f except for $x=0$.

④

a) Let $p(x)$ be a polynomial. $p(x) = a_n x^n + \dots + a_1 x + a_0$.

Then $\int_a^b f(x)p(x)\,dx = \int_a^b f(x)(a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0)\,dx$

$$= a_n \int_a^b f(x)x^n\,dx + a_{n-1}\int_a^b f(x)x^{n-1} + \dots + a_1 \int_a^b f(x)x\,dx + a_0 \int_a^b f(x)dx$$

$$= a_n \cdot 0 + a_{n-1}\cdot 0 + \dots + a_1 \cdot 0 + a_0 \cdot 0 = 0$$

b) By Weierstrass' Theorem, there exists polynomials $p_n(x)$ s.t. $p_n(x)$ converges uniformly to $f(x)$ on $[a,b]$.

Then $\int_a^b (f(x))^2 dx = \int_a^b f(x) \lim_{n\to\infty} p_n(x)dx = \lim_{n\to\infty} \int_a^b f(x)p_n(x)dx = \lim_{n\to\infty} 0 = 0.$

As $(f(x))^2 \geq 0$ on $[a,b]$, and $(f(x))^2$ continuous, the only way we can have $\int_a^b (f(x))^2 dx = 0$ is if $(f(x))^2 = 0$ on $[a,b] \Rightarrow f(x) = 0$ on $[a,b]$.

⑤

a) Choose any point $x \in X$. Need to show that given an $\varepsilon > 0$ we can find an $s \in B(x, \varepsilon)$, $s \in S$.
As T is dense in X, we know that there exists a $t \in B(x, \varepsilon/2)$, $t \in T$.
As S is dense in T, we know that there exists an $s \in B(t, \varepsilon/2)$, $s \in S$.
And then $d(x, s) \leq d(x,t) + d(t,s) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, so $s \in B(x, \varepsilon)$.

b) Let $c = \max\{|a|, |b|\}$, and let $p(x) = a_k x^k + a_{k-1} x^{k-1} + \dots + a_1 x + a_0$

As each $a_i \in \mathbb{R}$, we can find $a_{i,n} \in \mathbb{Q}$ s.t. $|a_i - a_{i,n}| < \frac{1}{nkc^i}$

Let $q_n(x) = a_{k,n} x^k + a_{k-1,n} x^{k-1} + \dots + a_{1,n} x + a_{0,n}$

Then
$$\begin{aligned} |p(x) - q_n(x)| &= |(a_k - a_{k,n}) x^k + \dots + (a_1 - a_{1,n}) x + (a_0 - a_{0,n})| \\ &\leq |a_k - a_{k,n}| |x|^k + \dots + |a_1 - a_{1,n}| |x| + |a_0 - a_{0,n}| \\ &< \frac{1}{nkc^k} c^k + \dots + \frac{1}{nkc} c + \frac{1}{nk} \\ &= \frac{k}{nk} = \frac{1}{n}. \end{aligned}$$

So $\rho(p, q_n) \leq \frac{1}{n} \to 0$ as $n \to \infty$, so q_n converges uniformly to p. And as all coefficients in q_n are rational numbers, we have what we want.

c) Let S be the set of polynomials with rational coefficients, and T be the set of polynomials.
For each $p \in T$, we have a sequence of $q_n \in S$, $q_n \to p$ in the ρ-metric.
So S is dense in T. And T is dense in $C([a,b], \mathbb{R})$ by Weierstrass' thm.
So by a), S is dense in $C([a,b], \mathbb{R})$, as wanted.

(4) cont.

d) Let $Q_0 = \mathbb{Q}$ be the set of all degree-zero polynomials with rational coefficients,

$Q_1 \approx \mathbb{Q} \times \mathbb{Q}$ be the set of all degree-one polynomials with rational coefficients,

$\vdots$

$Q_k \approx \mathbb{Q}^{k+1}$ be the set of all degree-k polynomials with rational coefficients.

$\vdots$

Then each Q_k is countable, as it is equivalent to a finite product of countable sets.

And then $S = \bigcup_{k=0}^{\infty} Q_k$, the set of all polynomials with rational coefficients must be countable, as it's a countable union of countable sets.

So S is countable, and it's dense in $C([a,b],\mathbb{R})$ by c),

and we have a countable dense subset.

5.11

(1)

Will check all three properties:

i) $\|x\| = \sqrt{x_1^2 + \dots + x_n^2} \geq 0$ as it's the root of a sum of non-negative numbers.

If $x = 0$, $x_i = 0$ for all i, so $\|x\| = \sqrt{0^2 + \dots + 0^2} = \sqrt{0} = 0$, and if $\|x\| = 0$, then $\sqrt{x_1^2 + \dots + x_n^2} = 0$. Then $x_1^2 + \dots + x_n^2 = 0 \Rightarrow x_1 = x_2 = \dots = x_n = 0$ as all $x_i^2 \geq 0$.

The same is true for $\|z\| = \sqrt{|z_1|^2 + \dots + |z_n|^2}$

ii) $\|\alpha x\| = \sqrt{(\alpha x_1)^2 + \dots + (\alpha x_n)^2} = \sqrt{\alpha^2 x_1^2 + \dots + \alpha^2 x_n^2} = \sqrt{\alpha^2}\sqrt{x_1^2 + \dots + x_n^2} = |\alpha|\,\|x\|$.

$\|\alpha z\| = \sqrt{|\alpha z_1|^2 + \dots + |\alpha z_n|^2} = \sqrt{|\alpha|^2|z_1|^2 + \dots + |\alpha|^2|z_n|^2} = \sqrt{|\alpha|^2}\sqrt{|z_1|^2 + \dots + |z_n|^2} = |\alpha|\,\|z\|$.

iii) The Cauchy-Schwarz-inequality on $\mathbb{R}^n$ gives us $\sum_{i=1}^{n} x_i y_i \leq \sqrt{\sum_{i=1}^{n} x_i^2}\sqrt{\sum_{i=1}^{n} y_i^2}$.

So $\|x+y\|^2 = (x_1+y_1)^2 + \dots + (x_n+y_n)^2 = x_1^2 + \dots + x_n^2 + y_1^2 + \dots + y_n^2 + 2\sum_{i=1}^{n} x_i y_i$

$= \|x\|^2 + \|y\|^2 + 2\sum_{i=1}^{n} x_i y_i \leq \|x\|^2 + \|y\|^2 + 2\|x\|\cdot\|y\|$

$= (\|x\| + \|y\|)^2$

So $\|x+y\| \leq \|x\| + \|y\|$.

For complex norm we have:

$\|z+w\|^2 = |z_1+w_1|^2 + \dots + |z_n+w_n|^2 \leq (|z_1|+|w_1|)^2 + \dots + (|z_n|+|w_n|)^2$

$= |z_1|^2 + \dots + |z_n|^2 + |w_1|^2 + \dots + |w_n|^2 + 2\sum_{i=1}^{n} |z_i||w_i|$

$\leq \|z\|^2 + \|w\|^2 + 2\cdot\|z\|\,\|w\| = (\|z\| + \|w\|)^2$

② Will check all the properties:

i) $\|f\| = \sup\{|f(x)| : x \in X\} \geq 0$ as it's the supremum over non-negative numbers.

If $f \equiv 0$, $\sup\{|f(x)| : x \in X\} = \sup\{0\} = 0$, and if $\sup\{|f(x)| : x \in X\} = 0$, $|f(x)| = 0$ for all x $\Rightarrow f \equiv 0$.

ii) $\|\alpha f\| = \sup\{|\alpha f(x)| : x \in X\} = \sup\{|\alpha||f(x)| : x \in X\}$
$= |\alpha| \sup\{|f(x)| : x \in X\} = |\alpha| \|f\|$.

iii) $\|f+g\| = \sup\{|f(x)+g(x)| : x \in X\} \leq \sup\{|f(x)| + |g(x)| : x \in X\}$
$\leq \sup\{|f(x)| : x \in X\} + \sup\{|g(x)| : x \in X\} = \|f\| + \|g\|$.

③ i) $\|f\|_1 = \int_a^b |f(x)| dx \geq 0$ as it's the integral of a positive function.

If $f \equiv 0$, $\int_a^b |0| dx = 0$, and if $\|f\| = 0$, $\int_a^b |f(x)| dx = 0 \Rightarrow |f(x)| = 0$ for all $x \in [a,b] \Rightarrow f \equiv 0$ on $[a,b]$.

ii) $\|\alpha f\| = \int_a^b |\alpha f(x)| dx = |\alpha| \int_a^b |f(x)| dx = |\alpha| \|f\|$.

iii) $\|f+g\| = \int_a^b |f(x)+g(x)| dx \leq \int_a^b |f(x)| + |g(x)| dx = \int_a^b |f(x)| dx + \int_a^b |g(x)| dx = \|f\| + \|g\|$.

④ Proof of 5.1.4.(ii), if $\{x_n\} \to x$, $\{y_n\} \to y$, then $\{x_n + y_n\} \to x+y$.

As $x_n \to x$, can choose $N_1 \in \mathbb{N}$ s.t. $\|x_n - x\| < \frac{\varepsilon}{2}$ when $n \geq N_1$, and as $y_n \to y$, can choose $N_2 \in \mathbb{N}$ s.t. $\|y_n - y\| < \frac{\varepsilon}{2}$ when $n \geq N_2$

Then $\|(x+y) - (x_n + y_n)\| = \|(x - x_n) + (y - y_n)\| \leq \|x - x_n\| + \|y - y_n\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

when $n \geq N = \max\{N_1, N_2\}$..

⑤ Need to show $\|u\| - \|v\| \leq \|u - v\|$ and $\|v\| - \|u\| \leq \|u - v\|$,

i.e. $\|u\| \leq \|u - v\| + \|v\|$ and $\|v\| \leq \|u - v\| + \|u\|$.

Have: $\|u\| = \|u - v + v\| \leq \|u - v\| + \|v\|$

and: $\|v\| = \|v - u + u\| \leq \|v - u\| + \|u\| = \|u - v\| + \|u\|$, as wanted.

⑥ Choose any vector $x \neq 0$. Assume $d(x,y) = \|x - y\|$ where d is the discrete metric. Then $\|x\| = d(x,0) = 1$, as $x \neq 0$. But $2x \neq x$ and $\|2x\| = |2|\|x\| = 2 \cdot 1 = 2$, and $d(2x, 0) = 1$, $2 \neq 1$. So cannot have $d(x,y) = \|x - y\|$ for any norm.

⑦ Let $\{x_n\}$ be a Cauchy sequence, $\|x_n - x_m\| < \varepsilon$ whenever $n, m \geq N$.

Let $a_n = \|x_n\|$ and $y_n = \frac{x_n}{a_n}$. Then $\|y_n\| = \left\|\frac{x_n}{a_n}\right\| = \frac{1}{a_n} \cdot \|x_n\| = 1$.

So $y_n \in S$.

Have $|a_n - a_m| = |\|x_n\| - \|x_m\|| \leq \|x_n - x_m\| < \varepsilon$ whenever $n, m \geq N$

So a_n is a Cauchy-sequence, and therefore bounded, $|a_n| < M$ for some $M \in \mathbb{R}$. Can then make sure that $\|x_n - x_m\| < \frac{M\varepsilon}{2}$ when $n, m \geq N'$

and then $\|y_n - y_m\| = \left\|\frac{x_n}{a_n} - \frac{x_m}{a_m}\right\| = \frac{1}{a_n}\left\|x_n - \frac{a_n}{a_m}x_m\right\| \leq \frac{1}{a_n}\left(\|x_n - x_m\| + \left\|x_m - \frac{a_n}{a_m}x_m\right\|\right)$

$$= \frac{1}{a_n}\left(\|x_n - x_m\| + \left|1 - \frac{a_n}{a_m}\right|\|x_m\|\right)$$

$$= \frac{1}{a_n}\left(\|x_n - x_m\| + |a_m - a_n|\right) \leq \frac{1}{M}\left(\frac{\varepsilon M}{2} + \frac{\varepsilon M}{2}\right) = \varepsilon$$

So both $\{a_n\}$ and $\{y_n\}$ are Cauchy.

c/.

(7) cont.

As $\{a_n\}$ is Cauchy, we have that $a_n \to a$, as $\mathbb{R}$ is complete.

As $\{y_n\}$ is Cauchy, we have that $y_n \to y$, as S is complete.

And then we have $\{x_n\} = \{a_n y_n\} \to ay = x$, by Prop. 5.1.4 (ii).

(8)

Want to show that $\|\bullet\|_1$ is equivalent to $\|\bullet\|_2$, if we know that $\|\bullet\|_1$ is equivalent to $\|\cdot\|$ and $\|\bullet\|_2$ is equivalent to $\|\bullet\|$, where $\|\cdot\|$ is the euclidean norm.

As $\|\bullet\|_1$ is equivalent to $\|\cdot\|$, we have M_1 and K_1 s.t.

$$\|x\|_1 \le M_1 \|x\| \quad \text{and} \quad \|x\| \le K_1 \|x\|_1 \quad \text{for all } x \in \mathbb{R}^n$$

As $\|\bullet\|_2$ is equivalent to $\|\bullet\|$, we have M_2 and K_2 s.t.

$$\|x\|_2 \le M_2 \|x\| \quad \text{and} \quad \|x\| \le K_2 \|x\|_2 .$$

Then:

$$\|x\|_1 \le M_1 \|x\| \le M_1 K_2 \|x\|_2 \quad \text{and} \quad \|x\|_2 \le M_2 \|x\| \le M_2 K_1 \|x\|_1$$

so $\|x\|_1 \le M \|x\|_2$ and $\|x\|_2 \le K \|x\|_1$ with $M = M_1 K_2$ and $K = M_2 K_1$, and we have that $\|\bullet\|_1$ and $\|\bullet\|_2$ are equivalent.

(9) Check that $V = V_1 \times \dots \times V_n$ is a linear space: (Def 5.1.1)

i) $u + v = (u_1, \dots, u_n) + (v_1, \dots, v_n) = (u_1 + v_1, \dots, u_n + v_n)$
$= (v_1 + u_1, \dots, v_n + u_n) = (v_1, \dots, v_n) + (u_1, \dots, u_n) = v + u.$

ii) $(u+v) + w = ((u_1, \dots, u_n) + (v_1, \dots, v_n)) + (w_1, \dots, w_n)$
$= (u_1 + v_1, \dots, u_n + v_n) + (w_1, \dots, w_n)$
$= ((u_1 + v_1) + w_1, \dots, (u_n + v_n) + w_n) = (u_1 + (v_1 + w_1), \dots, u_n + (v_n + w_n))$
$= (u_1, \dots, u_n) + (v_1 + w_1, \dots, v_n + w_n)$
$= u + ((v_1, \dots, v_n) + (w_1, \dots, w_n)) = u + (v + w).$

iii) In each V_i there exists a 0_i s.t. $u_i + 0_i = u_i$.
Choose $0 = (0_1, \dots, 0_n)$.
Then $u + 0 = (u_1, \dots, u_n) + (0_1, \dots, 0_n) = (u_1 + 0_1, \dots, u_n + 0_n)$
$= (u_1, \dots, u_n) = u.$ for all $u \in V$.

iv) In each V_i, foreach u_i there exists a $-u_i$ s.t. $u + (-u_i) = 0_i$
Given $u \in V$, $u = (u_1, \dots, u_n)$, let $-u = (-u_1, \dots, -u_n)$
Then $u + (-u) = (u_1, \dots, u_n) + (-u_1, \dots, -u_n) = (u_1 + (-u_1), \dots, u_n + (-u_n))$
$= (0_1, \dots, 0_n) = 0.$

v) $\alpha(u+v) = \alpha((u_1, \dots, u_n) + (v_1, \dots, v_n)) = \alpha(u_1 + v_1, \dots, u_n + v_n)$
$= (\alpha(u_1 + v_1), \dots, \alpha(u_n + v_n)) = (\alpha u_1 + \alpha v_1, \dots, \alpha u_n + \alpha v_n)$
$= (\alpha u_1, \dots, \alpha u_n) + (\alpha v_1, \dots, \alpha v_n)$
$= \alpha(u_1, \dots, u_n) + \alpha(v_1, \dots, v_n) = \alpha u + \alpha v$

vi) $(\alpha+\beta)u = (\alpha+\beta)(u_1, \dots, u_n) = ((\alpha+\beta)u_1, \dots, (\alpha+\beta)u_n)$

$= (\alpha u_1 + \beta u_1, \dots, \alpha u_n + \beta u_n) = (\alpha u_1, \dots, \alpha u_n) + (\beta u_1, \dots, \beta u_n)$

$= \alpha(u_1, \dots, u_n) + \beta(u_1, \dots, u_n) = \alpha u + \beta u$

vii) $\alpha(\beta u) = \alpha(\beta(u_1, \dots, u_n)) = \alpha(\beta u_1, \dots, \beta u_n) = (\alpha(\beta u_1), \dots, \alpha(\beta u_n))$

$= ((\alpha\beta)u_1, \dots, (\alpha\beta)u_n) = (\alpha\beta)(u_1, \dots, u_n)$

viii) $1u = 1(u_1, \dots, u_n) = (1u_1, \dots, 1u_n) = (u_1, \dots, u_n) = u$

So V is a linear space. Now to show that $\|\cdot\|$ is a norm. (Def 5.1.2)

i) $\|u\| = \|(u_1, \dots, u_n)\| = \|u_1\|_1 + \|u_2\|_2 + \dots + \|u_n\|_n \geq 0$ as $\|u_i\|_i \geq 0$ for all i

If $\|u\| = 0$, then $\|u_i\|_i = 0$ for all i, so $u_i = 0_i$. Then we have $u = (0_1, \dots, 0_n) = 0$.

ii) $\|\alpha u\| = \|\alpha(u_1, \dots, u_n)\| = \|(\alpha u_1, \dots, \alpha u_n)\| = \|\alpha u_1\|_1 + \dots + \|\alpha u_n\|_n$

$= |\alpha|\|u_1\|_1 + \dots + |\alpha|\|u_n\|_n = |\alpha|(\|u_1\|_1 + \dots + \|u_n\|_n) = |\alpha|\|u\|$

iii) $\|u+v\| = \|(u_1, \dots, u_n) + (v_1, \dots, v_n)\| = \|(u_1+v_1, \dots, u_n+v_n)\|$

$= \|u_1+v_1\|_1 + \dots + \|u_n+v_n\|_n \leq \|u_1\|_1 + \|v_1\|_1 + \dots + \|u_n\|_n + \|v_n\|_n$

$= \|u_1\|_1 + \dots + \|u_n\|_n + \|v_1\|_1 + \dots + \|v_n\|_n = \|u\| + \|v\|$.

So $\|\cdot\|$ is a norm, and $(V, \|\cdot\|)$ is a normed linear space.

⑩

Choose a Cauchy sequence $\{u^k\}$, $u^k \in V$.

Write each $u^k = (u_1^k, u_2^k, \dots, u_n^k)$.

As $\{u^k\}$ is Cauchy, we have that given an $\varepsilon > 0$, there exists $N \in \mathbb{N}$ s.t. $\|u^k - u^l\| < \varepsilon$ whenever $k, l \geq N$.

We have $\|u^k - u^l\| = \|(u_1^k, \dots, u_n^k) - (u_1^l, \dots, u_n^l)\| = \|(u_1^k - u_1^l, \dots, u_n^k - u_n^l)\|$

$$= \|u_1^k - u_1^l\|_1 + \dots + \|u_n^k - u_n^l\|_n < \varepsilon.$$

So $\|u_i^k - u_i^l\| < \varepsilon$ whenever $k, l \geq N$, i.e. $\{u_i^k\}_{k=0}^{\infty}$ is Cauchy.

As each V_i is complete, each $\{u_i^k\}$ must converge to a point in V_i, call it u_i.

Now choose $u = (u_1, \dots, u_n) \in V$. Want to show that $\{u^k\}$ converges to u.

As each $\{u_i^k\}$ converges to u_i, we can for any $\varepsilon > 0$ find an N_i s.t. $\|u_i^k - u_i\|_i < \frac{\varepsilon}{n}$ whenever $k \geq N_i$. Choose $M = \max_{1 \leq i \leq n} \{N_i\}$.

Then $\|u^k - u\| = \|(u_1^k, \dots, u_n^k) - (u_1, \dots, u_n)\| = \|(u_1^k - u_1, \dots, u_n^k - u_n)\|$

$$= \|u_1^k - u_1\|_1 + \dots + \|u_n^k - u_n\| < \frac{\varepsilon}{n} + \dots + \frac{\varepsilon}{n} = \varepsilon$$

whenever $k \geq M$, so u^k converges to u, and V must be complete.

(11)

a) T is linear $\iff T(\alpha X + \beta Y) = \alpha T(X) + \beta T(Y)$

Let $X = (x_1, \dots, x_n)$ and $Y = (y_1, \dots, y_n)$. Then $\alpha X + \beta Y = (\alpha x_1 + \beta y_1, \dots, \alpha x_n + \beta y_n)$, and $T(\alpha X + \beta Y) = T(\alpha x_1 + \beta y_1, \dots, \alpha x_n + \beta y_n)$

$$= (\alpha x_1 + \beta y_1) e_1 + \dots + (\alpha x_n + \beta y_n) e_n$$
$$= \alpha x_1 e_1 + \beta y_1 e_1 + \dots + \alpha x_n e_n + \beta y_n e_n$$
$$= \alpha (x_1 e_1 + x_2 e_2 + \dots + x_n e_n) + \beta (y_1 e_1 + \dots + y_n e_n)$$
$$= \alpha T(x_1, \dots, x_n) + \beta T(y_1, \dots, y_n) = \alpha T(X) + \beta T(Y).$$

T is bijective:

T is surjective:

Let v be any vector in V. We can then write $v = a_1 e_1 + \dots + a_n e_n$ by the definition of a basis. Then $v = T(a_1, \dots, a_n)$, so T must be surjective.

T is injective:

Let X, Y be two different vectors in $\mathbb{R}^n$, and assume $T(X) = T(Y)$.

Then $x_1 e_1 + x_2 e_2 + \dots + x_n e_n = y_1 e_1 + y_2 e_2 + \dots + y_n e_n$, but by the definition of a basis, we must have $x_1 = y_1, x_2 = y_2, \dots, x_n = y_n$. So we have $X = Y$, a contradiction, and T must be injective.

b) Must check definition of norm:

i) $\|x\|_1 = \|T(x)\| \geq 0$ as $\|\cdot\|$ is a norm.

If $\|x\|_1 = 0$, then $\|T(x)\| = 0 \iff T(x) = 0 \iff x_1 e_1 + \dots + x_n e_n = 0$

By definition of a basis, this implies $x_1 = x_2 = x_3 = \dots = x_n = 0$, so $x = 0$.

ii) $\|\alpha x\|_1 = \|T(\alpha x)\| = \|\alpha T(x)\| = |\alpha| \|T(x)\| = |\alpha| \|x\|_1$, where we used that T was linear.

iii) $\|x+y\|_1 = \|T(x+y)\| = \|T(x) + T(y)\| \leq \|T(x)\| + \|T(y)\| = \|x\|_1 + \|y\|_1$, where we used that T was linear.

(11) cont.

c) As T is bijective, the inverse function $T^{-1}: V \to \mathbb{R}^n$ exists.

From b) we have that if v is a vector in V, we have that

$\|v\| = \|T^{-1}(v)\|_1$, for a norm $\|\cdot\|$ on V.

Now choose another norm on V, $\widehat{\|\cdot\|}$, and construct the corresponding norm on $\mathbb{R}^n$, $\|\cdot\|_2$ by $\|\hat{v}\| = \|T^{-1}(v)\|_2$. As all norms on $\mathbb{R}^n$ are equivalent, there exists M, K, s.t. $\|x\|_1 \le M\|x\|_2$ and $\|x\|_2 \le K\|x\|_1$.

But then we have

$$\|v\| = \|T^{-1}(v)\|_1 \le M\|T^{-1}(v)\|_2 = M\|\hat{v}\|$$

and $\|\hat{v}\| = \|T^{-1}(v)\|_2 \le K\|T^{-1}(v)\|_1 = K\|v\|$, so $\|\cdot\|$ and $\widehat{\|\cdot\|}$ must be equivalent.

This is true for any two norms, so all norms must be equivalent.