5.3

(8) Must show that $|\langle u_n, v_n\rangle - \langle u, v\rangle| < \varepsilon$ for $n \geq N$.

We have $|\langle u_n, v_n\rangle - \langle u, v\rangle| = |\langle u_n, v_n - v + v\rangle - \langle u, v\rangle|$

$= |\langle u_n, v_n - v\rangle + \langle u_n - u, v\rangle|$ Triangle ineq.

$\leq |\langle u_n, v_n - v\rangle| + \langle u_n - u, v\rangle|$ Cauchy-Schwarz.

$\leq \|u_n\|\|v_n - v\| + \|u_n - u\|\|v\|$

As $u_n$ converges, $\|u_n\|$ is bounded by an $M$, and we can find $N_1$ s.t. $\|u_n - u\| < \frac{\varepsilon}{2\|v\|}$ for $n \geq N_1$.

As $v_n$ converges, we can find an $N$ s.t. $\|v_n - v\| < \frac{\varepsilon}{2M}$.

Then $|\langle u_n, v_n\rangle - \langle u, v\rangle| < M\frac{\varepsilon}{2M} + \frac{\varepsilon}{2\|v\|}\|v\| = \varepsilon$ as wanted.

(10)

$$P(\alpha u) = \langle \alpha u, e_1\rangle e_1 + \dots + \langle \alpha u, e_n\rangle e_n$$
$$= \alpha\langle u, e_1\rangle e_1 + \dots + \alpha\langle u, e_n\rangle e_n$$
$$= \alpha(\langle u, e_1\rangle e_1 + \dots + \langle u, e_n\rangle e_n)$$
$$= \alpha P(u).$$

$$P(u+v) = \langle u+v, e_1\rangle + \dots + \langle u+v, e_n\rangle e_n$$
$$= (\langle u, e_1\rangle + \langle v, e_1\rangle) e_1 + \dots + (\langle u, e_n\rangle + \langle v, e_n\rangle) e_n$$
$$= \langle u, e_1\rangle e_1 + \dots + \langle u, e_n\rangle e_n + \langle v, e_1\rangle e_1 + \dots + \langle v, e_n\rangle e_n$$
$$= P(u) + P(v).$$

12

a) Choose a sequence of $\{s_n\} \subseteq S^{\perp}$ converging to a $v \in V$. If $v \in S^{\perp}$, we have that $S^{\perp}$ must be closed.

By 8, $\langle s_n, s\rangle \Rightarrow \langle v, s\rangle$ for all $s \in S$.

As $\langle s_n, s\rangle = 0$ for all $n \in \mathbb{N}$, for all $s \in S$, we must have $\langle v, s\rangle = 0$ for all $s \in S$, so $v \in S^{\perp}$.

b) Let $t \in T^{\perp}$. Want to show $t \in S^{\perp}$.

As $t \in T^{\perp}$, $\langle t, s\rangle = 0$ for all $s \in T$. As $S \subseteq T$, we then have that $\langle t, s\rangle = 0$ for all $s \in S$. Therefore, $t \in S^{\perp}$, as wanted.

13 a) Let $\langle x, y\rangle_N$ denote the standard inner product on $\mathbb{R}^N$. Before we use the hint from the exercise, let us define

$$x_N^+ = (|x_1|, \dots, |x_N|) \quad \text{and} \quad y_N^+ = (|y_1|, \dots, |y_N|)$$

Then, by Cauchy-Schwarz, we have

$$\sum_{n=1}^{N} |x_n||y_n| = \langle x_N^+, y_N^+\rangle_N \leq \|x_N^+\|\|y_N^+\| = \left(\sum_{n=1}^{N} |x_n|^2\right)^{1/2}\left(\sum_{n=1}^{N} |y_n|^2\right)^{1/2}$$

$$= \left(\sum_{n=1}^{N} x_n^2\right)^{1/2}\left(\sum_{n=1}^{N} y_n^2\right)^{1/2}$$

$$\leq \left(\sum_{n=1}^{\infty} x_n^2\right)^{1/2}\left(\sum_{n=1}^{\infty} y_n^2\right)^{1/2}$$

for all $N \in \mathbb{N}$.

As this is true for all $N \in \mathbb{N}$, we have

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \left(\sum_{n=1}^{\infty} x_n^2\right)^{1/2}\left(\sum_{n=1}^{\infty} y_n^2\right),$$

and as $\sum_{n=1}^{\infty} |x_n y_n|$ is a bounded series of positive elements, $\sum_{n=1}^{\infty} |x_n y_n|$ must converge as well.

Then $\sum_{n=1}^{\infty} x_n y_n$ is absolutely convergent, and must converge as well.

13

b) We still define $\bar{x}+\bar{y}=\{x_n\}+\{y_n\}=\{x_n+y_n\}$
and $\alpha\bar{x}=\alpha\{x_n\}=\{\alpha x_n\}$.

$\bar{x}$ here means that $\bar{x}$ is a vector, not that we take the complex conjugate.

Check first:

If $\bar{x},\bar{y}\in l_2$, is $\bar{x}+\bar{y}\in l_2$?

$$\sum_{n=1}^{\infty}(x_n+y_n)^2=\sum_{n=1}^{\infty}x_n^2+2x_ny_n+y_n^2=\sum_{n=1}^{\infty}x_n^2+2\sum_{n=1}^{\infty}x_ny_n+\sum_{n=1}^{\infty}y_n^2$$

As these all converge (one of them by a)), we have that $\sum_{n=1}^{\infty}(x_n+y_n)^2$ converge, so $\bar{x}+\bar{y}\in l_2$.

If $\bar{x}\in l_2$, is $\alpha\bar{x}\in l_2$?

$$\sum_{n=1}^{\infty}(\alpha x_n)^2=\sum_{n=1}^{\infty}\alpha^2x_n^2=\alpha^2\sum_{n=1}^{\infty}x_n^2$$ which converges, so $\alpha\bar{x}\in l_2$.

Now we must check all the properties of a vector space.

i) $\bar{x}+\bar{y}=\bar{y}+\bar{x}$:

$\bar{x}+\bar{y}=\{x_n\}+\{y_n\}=\{x_n+y_n\}=\{y_n+x_n\}=\bar{y}+\bar{x}$

ii) $(\bar{x}+\bar{y})+\bar{z}=\bar{x}+(\bar{y}+\bar{z})$

$(\bar{x}+\bar{y})+\bar{z}=(\{x_n\}+\{y_n\})+\{z_n\}=\{x_n+y_n\}+\{z_n\}$

$=\{(x_n+y_n)+z_n\}=\{x_n+(y_n+z_n)\}=\{x_n\}+\{y_n+z_n\}$

$=\bar{x}+(\{y_n\}+\{z_n\})=\bar{x}+(\bar{y}+\bar{z})$

iii) Choose $\bar{0}=\{0\}_{n\in\mathbb{N}}$. Then $\sum_{n=1}^{\infty}0^2=0<\infty$, so $\bar{0}\in l_2$.

We have $\bar{x}+\bar{0}=\{x_n\}+\{0\}=\{x_n+0\}=\{x_n\}=\bar{x}$, as wanted.

iv) Given $\bar{x}=\{x_n\}\in l_2$, let $-\bar{x}=\{-x_n\}$. Then $\sum_{n=1}^{\infty}(-x_n)^2=\sum_{n=1}^{\infty}x_n^2<\infty$, so $-\bar{x}\in l_2$. We have $\bar{x}+(-\bar{x})=\{x_n\}+\{-x_n\}=\{x_n+(-x_n)\}=\{0\}=\bar{0}$, as wanted.

v) $\alpha(\bar{x}+\bar{y})=\alpha(\{x_n\}+\{y_n\})=\alpha\{x_n+y_n\}=\{\alpha(x_n+y_n)\}=\{\alpha x_n+\alpha y_n\}$

$=\{\alpha x_n\}+\{\alpha y_n\}=\alpha\{x_n\}+\alpha\{y_n\}=\alpha\bar{x}+\alpha\bar{y}$, as wanted.

vi) $(\alpha+\beta)\bar{x}=(\alpha+\beta)\{x_n\}=\{(\alpha+\beta)x_n\}=\{\alpha x_n+\beta x_n\}=\{\alpha x_n\}+\{\beta x_n\}$

$=\alpha\{x_n\}+\beta\{x_n\}=\alpha\bar{x}+\beta\bar{x}$.

vii) $\alpha(\beta\bar{x})=\alpha(\beta\{x_n\})=\alpha\{\beta x_n\}=\{\alpha(\beta x_n)\}=\{(\alpha\beta)x_n\}=(\alpha\beta)\{x_n\}$

$=(\alpha\beta)\bar{x}$ as wanted.

viii) $1\cdot\bar{x}=1\cdot\{x_n\}=\{1\cdot x_n\}=\{x_n\}=\bar{x}$

$\square$

13

c) We know that $\sum_{n=1}^{\infty} x_n y_n$ converges, so $\langle \cdot, \cdot \rangle$ is a function with values in $\mathbb{R}$. Check the properties of an inner product.

i) $\langle \bar{x}, \bar{y} \rangle = \langle \bar{y}, \bar{x} \rangle$:

$$\langle \bar{x}, \bar{y} \rangle = \sum_{n=1}^{\infty} x_n y_n = \sum_{n=1}^{\infty} y_n x_n = \langle \bar{y}, \bar{x} \rangle \quad \text{as wanted.}$$

ii) $\langle \alpha \bar{x}, \bar{y} \rangle = \alpha \langle \bar{x}, \bar{y} \rangle$:

$$\langle \alpha \bar{x}, \bar{y} \rangle = \sum_{n=1}^{\infty} (\alpha x_n) y_n = \sum_{n=1}^{\infty} \alpha x_n y_n = \alpha \sum_{n=1}^{\infty} x_n y_n = \alpha \langle \bar{x}, \bar{y} \rangle$$

iii) For all $\bar{x} \in l_2$, we have $\langle \bar{x}, \bar{x} \rangle \geq 0$, equality iff. $\bar{x} = \bar{0}$.

We have $\langle \bar{x}, \bar{x} \rangle = \sum_{n=1}^{\infty} x_n^2 \geq 0$ as it's a sum of non-negative numbers.

If $\bar{x} = \bar{0}$, we have $\langle \bar{x}, \bar{x} \rangle = 0$, and if $\langle \bar{x}, \bar{x} \rangle = 0$, $\sum_{n=1}^{\infty} x_n^2 = 0$, which must mean $x_n = 0$ for all $n$, so $\bar{x} = \bar{0}$.

d) Choose $\{\bar{x}^n\}$ a Cauchy-sequence in $l_2$. Must show that it converges.

As $\{\bar{x}^n\}$ is Cauchy, we have $\|\bar{x}^n - \bar{x}^m\| = \sum_{k=1}^{\infty} (x_k^n - x_k^m)^2 < \varepsilon^2$ when $n, m \geq N$.

Therefore, we have that $(x_k^n - x_k^m)^2 < \varepsilon^2$ for $n, m \geq N$, and a fixed $k$.

This implies $|x_k^n - x_k^m| < \varepsilon$ when $n, m \geq N$, so $\{x_k^n\}_{n \in \mathbb{N}}$ is a Cauchy-sequence in $\mathbb{R}$. As $\mathbb{R}$ is complete, $x_k^n$ must converge to a number $x_k$.

Now let $\bar{x} = \{x_k\}_{k \in \mathbb{N}}$. Want to show that $\bar{x} \in l_2$ and that $\bar{x}^n \to \bar{x}$.

As $\{\bar{x}^n\}$ is a Cauchy sequence, it is bounded, $\|\bar{x}^n\| \leq M$ for all $n \in \mathbb{N}$.

Then, for all $K \in \mathbb{N}$, we have $\sum_{k=1}^{K} x_k^2 = \sum_{k=1}^{K} \lim_{n \to \infty} (x_k^n)^2 = \lim_{n \to \infty} \sum_{k=1}^{K} (x_k^n)^2$

$$\leq \lim_{n \to \infty} \sum_{k=1}^{\infty} (x_k^n)^2 = \lim_{n \to \infty} \|\bar{x}^n\| \leq \lim_{n \to \infty} M = M$$

As this is true for <u>all</u> $K$, we must have $\sum_{k=1}^{\infty} x_k \leq M < \infty$, so $\bar{x} \in l_2$.

For all $K \in \mathbb{N}$, we also have $\sum_{k=1}^{K} (x_k^n - x_k^m)^2 \leq \sum_{k=1}^{\infty} (x_k^n - x_k^m)^2 = \|\bar{x}^n - \bar{x}^m\| < \frac{\varepsilon}{2}$ when $n, m \geq N'$

So $\sum_{k=1}^{K} (x_k^n - x_k)^2 = \sum_{k=1}^{K} \lim_{m \to \infty} (x_k^n - x_k^m)^2 = \lim_{m \to \infty} \sum_{k=1}^{K} (x_k^n - x_k^m)^2 \leq \frac{\varepsilon}{2}$ when $n \geq N'$

As this is true for <u>all</u> $K$, we have $\sum_{k=1}^{\infty} (x_k^n - x_k)^2 = \|\bar{x}^n - \bar{x}\| \leq \frac{\varepsilon}{2} < \varepsilon$ when $n \geq N'$,

so $\bar{x}^n \to \bar{x}$ as wanted.

(13)

(e) $\bar{e}_n = \{e_n^k\}$ s.t. $e_n^k = 1$ if $k=n$, and $e_n^k = 0$ if $k \neq n$.

To check that $\bar{e}_n$ is an orthonormal basis we have to check three things.

① $\langle \bar{e}_n, \bar{e}_m \rangle = 0$ if $n \neq m$ and $1$ if $n=m$, i.e. $\{\bar{e}_n\}$ is orthonormal.

② For each $\bar{x} \in l_2$, there exists an $\{a_n\}$ s.t. $\bar{x} = \sum_{n=1}^{\infty} a_n \bar{e}_n$

③ This $\{a_n\}$ is unique, i.e. if $\bar{x} = \sum_{n=1}^{\infty} b_n \bar{e}_n$, then $\{b_n\} = \{a_n\}$.

① $\langle \bar{e}_n, \bar{e}_m \rangle = \sum_{k=1}^{\infty} e_n^k e_m^k = e_n^n e_m^n + e_m^n e_m^m = 1 \cdot 0 + 0 \cdot 1 = 0$ if $n \neq m$

$\langle \bar{e}_n, \bar{e}_n \rangle = \sum_{k=1}^{\infty} e_n^k e_n^k = \sum_{k=1}^{\infty} (e_n^k)^2 = 1 \cdot 1 = 1$

② By Parseval's theorem, $\{a_n\}$ should be given by $\langle \bar{x}, \bar{e}_n \rangle$

$\langle \bar{x}, \bar{e}_n \rangle = \sum_{k=1}^{\infty} x_k e_n^k = x_n e_n^n = x_n$, so $\{a_n\} = \{x_n\}$.

Then $\|\bar{x} - \sum_{n=1}^{N} x_n \bar{e}_n\| = \sum_{n=1}^{N} (x_n - x_n)^2 + \sum_{n=N+1}^{\infty} x_n^2 = \sum_{n=N+1}^{\infty} x_n^2$

As $\sum_{n=1}^{\infty} x_n^2$ converges, we can get $\sum_{n=N+1}^{\infty} x_n^2 < \varepsilon$ for $N$ large enough,

so $\|\bar{x} - \sum_{n=1}^{N} x_n \bar{e}_n\| < \varepsilon$ when $N$ large enough, i.e. $\bar{x} = \sum_{n=1}^{\infty} x_n \bar{e}_n$.

③ Assume $\{b_n\} \neq \{x_n\}$. Will show $\sum b_n \bar{e}_n \neq \bar{x}$.

As $\{b_n\} \neq \{x_n\}$, there exists a $k$ s.t. $b_k \neq x_k$, $|x_k - b_k| > 0$.

Choose $\varepsilon$ s.t. $\sqrt{\varepsilon} < |x_k - b_k|$. Then, for $N > k$, we have

$\|\bar{x} - \sum_{n=1}^{N} b_n \bar{e}_n\| = \sum_{n=1}^{N} (x_n - b_n)^2 + \sum_{n=N+1}^{\infty} x_n^2 \geq (x_k - b_k)^2 > \sqrt{\varepsilon}^2 = \varepsilon$.

So $\sum_{n=1}^{N} b_n \bar{e}_n$ cannot converge to $\bar{x}$, and $\{a_n\}$ is therefore unique.

(13)

f) There are two ways to do this exercise, the "straightforward" approach and the object recognition approach, but they amount to essentially the same.

Straightforward:

Choose a Cauchy sequence $\bar{u}_n$ in $V$. Then $\|\bar{u}_n - \bar{u}_m\| < \varepsilon$ for $n, m \geq N$.

Let $a_n^i = \langle \bar{u}_n, \bar{v}_i \rangle$. By Parseval's theorem, we have that

(*) $\|\bar{u}_n\|^2 = \sum_{i=1}^{\infty} \langle \bar{u}_n, \bar{v}_i \rangle^2 = \sum_{i=1}^{\infty} (a_n^i)^2$, so $\bar{a}_n = \{a_n^i\}_{i \in \mathbb{N}} \in l_2$.

Again, by Parseval's theorem, we have that

(**) $\|\bar{u}_n - \bar{u}_m\|^2 = \sum_{i=1}^{\infty} \langle \bar{u}_n - \bar{u}_m, \bar{v}_i \rangle^2 = \sum_{i=1}^{\infty} (\langle \bar{u}_n, \bar{v}_i \rangle - \langle \bar{u}_m, \bar{v}_i \rangle)^2 = \sum_{i=1}^{\infty} (a_n^i - a_m^i)^2 = \|\bar{a}_n - \bar{a}_m\|^2$

So as $\|\bar{u}_n - \bar{u}_m\| < \varepsilon$ when $n, m \geq N$, we have $\|\bar{a}_n - \bar{a}_m\| < \varepsilon$ when $n, m \geq N$, and $\{\bar{a}_n\}_{n \in \mathbb{N}}$ must be Cauchy in $l_2$. By d), $l_2$ is complete, so $\bar{a}_n \to \bar{a} = \{a^i\}$. By the assumption on $V$, there exists an $\bar{u}$ s.t. $\langle \bar{u}, \bar{v}_i \rangle = a^i$. Must show $\bar{u}_n \to \bar{u}$.

We have (again, Parseval's thm), $\|\bar{u} - \bar{u}_n\|^2 = \sum_{i=1}^{\infty} \langle u - u_n, v_i \rangle^2 = \sum_{i=1}^{\infty} (\langle u, v_i \rangle - \langle u_n, v_i \rangle)^2$

$= \sum_{i=1}^{\infty} (a^i - a_n^i)^2 = \|\bar{a} - \bar{a}_n\|^2$.

So as $\bar{a}_n \to \bar{a}$, we have $\|\bar{a} - \bar{a}_n\| < \varepsilon$ when $n \geq N'$, so we get $\|\bar{u} - \bar{u}_n\| < \varepsilon$ when $n \geq N'$, and $\bar{u}_n \to \bar{u}$.

Object recognition: $V$ and $l_2$ are "essentially" the same object.

Let $f: V \to l_2$ be given by $f(\bar{u}) = \{\langle \bar{u}, \bar{v}_i \rangle\}$.

By (*), $\{\langle \bar{u}, \bar{v}_i \rangle\}$ is a square summable sequence, so this is well defined.

By (**) $f$ preserves distance, $\|\bar{u} - \bar{v}\| = \|f(\bar{u}) - f(\bar{v})\|$.

$f$ is injective: Given $\bar{u}, \bar{v} \in V$, $f(\bar{u}) = f(\bar{v}) = \{a_i\}$.

By Parseval's Thm, $\bar{u} = \sum \langle \bar{u}, \bar{v}_i \rangle \bar{v}_i = \sum a_i v_i = \sum \langle \bar{v}, \bar{v}_i \rangle \bar{v}_i = \bar{v}$, so $\bar{u} = \bar{v}$.

$f$ is surjective: Given $\{a_i\} \in l_2$, by the assumption on $V$, we have an $\bar{u} \in V$ s.t. $\langle u, v_i \rangle = a_i$, i.e. $f(\bar{u}) = \{a_i\}$.

So $f$ is a bijective distance-preserving function between metric spaces, i.e. an isometry (from 3.1). Isometries preserve metric properties, so if two spaces are isometric and one of them is complete, the other must be complete as well. So as $l_2$ is complete, we have that $V$ is complete.

7.1

① For all $f, g \in C([-\pi,\pi], \mathbb{C})$, we have that the integral $\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx$ exists and gives a complex number, so this is well defined.

Check properties for inner product:

i) $\langle f,g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{\overline{f(x)}g(x)}\,dx = \overline{\frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\overline{f(x)}\,dx} = \overline{\langle g,f\rangle}$

ii) $\langle f+g,h\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} (f(x)+g(x))\overline{h(x)}\,dx = \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} f(x)\overline{h(x)}\,dx + \int_{-\pi}^{\pi} g(x)\overline{h(x)}\,dx\right)$

$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{h(x)}\,dx + \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\overline{h(x)}\,dx = \langle f,h\rangle + \langle g,h\rangle$

iii) $\langle \alpha f,g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\alpha f(x))\overline{g(x)}\,dx = \frac{\alpha}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx = \alpha\langle f,g\rangle$

iv) $\langle f,f\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{f(x)}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx \geq 0$

as it's the integral over a non-negative function.

As $f$ is continuous, this integral can only be zero if $f(x)=0$ for all $x \in [-\pi,\pi]$, as wanted.

② $e^{i(x+y)} = \cos(x+y) + i\sin(x+y)$

$e^{i(x+y)} = e^{ix}e^{iy} = (\cos x + i\sin x)(\cos y + i\sin y) = \cos x\cos y - \sin x\sin y + i\cos x\sin y + i\sin x\cos y$.

Equate real and imaginary parts to get

$\cos(x+y) = \cos x\cos y - \sin x\sin y$ and $\sin(x+y) = \cos x\sin y + \sin x\cos y$

as wanted.

③ a) $\sin u \sin v = \frac{e^{iu}-e^{-iu}}{2i} \cdot \frac{e^{iv}-e^{-iv}}{2i} = \frac{e^{iu}e^{iv}-e^{iu}e^{-iv}-e^{-iu}e^{iv}+e^{-iu}e^{-iv}}{-4}$

$= \frac{e^{i(u+v)}-e^{i(u-v)}-e^{-i(u-v)}+e^{-i(u+v)}}{-4}$

$= \frac{e^{i(u-v)}+e^{-i(u-v)}}{2\cdot 2} - \frac{e^{i(u+v)}+e^{-i(u+v)}}{2\cdot 2}$

$= \frac{1}{2}\cos(u-v) - \frac{1}{2}\cos(u+v)$ as wanted.

b) $\int \sin 4x \sin x \, dx = \int \frac{1}{2}\cos(4x-x) - \frac{1}{2}\cos(4x+x)\, dx$

$= \frac{1}{2}\int \cos 3x \, dx - \frac{1}{2}\int \cos 5x \, dx$

$= \frac{1}{6}\sin 3x - \frac{1}{10}\sin 5x + C$

c) $\cos u \cos v = \frac{e^{iu}+e^{-iu}}{2} \cdot \frac{e^{iv}+e^{-iv}}{2} = \frac{e^{iu}e^{iv}+e^{iu}e^{-iv}+e^{-iu}e^{iv}+e^{-iu}e^{-iv}}{2\cdot 2}$

$= \frac{e^{i(u+v)}+e^{-i(u+v)}+e^{i(u-v)}+e^{-i(u-v)}}{2\cdot 2}$

$= \frac{1}{2}\frac{e^{i(u+v)}+e^{-i(u+v)}}{2} + \frac{1}{2}\frac{e^{i(u-v)}+e^{-i(u-v)}}{2}$

$= \frac{1}{2}\cos(u+v) + \frac{1}{2}\cos(u-v)$

$\int \cos 3x \cos 2x \, dx = \int \frac{1}{2}\cos(3x+2x) + \frac{1}{2}\cos(3x-2x)\, dx$

$= \frac{1}{2}\int \cos 5x \, dx + \frac{1}{2}\int \cos x \, dx = \frac{1}{10}\sin 5x + \frac{1}{2}\sin x + C$

③ d)

$$\sin u \cos v = \frac{e^{iu}-e^{-iu}}{2i}\,\frac{e^{iv}+e^{-iv}}{2} = \frac{e^{iu}e^{iv}+e^{iu}e^{-iv}-e^{-iu}e^{iv}-e^{-iu}e^{-iv}}{4i}$$

$$= \frac{e^{i(u+v)}-e^{-i(u+v)}}{2\cdot 2i} + \frac{e^{i(u-v)}-e^{-i(u-v)}}{2\cdot 2i} = \frac{1}{2}\sin(u+v)+\frac{1}{2}\sin(u-v)$$

$$\int \sin x \cos 4x\,dx = \int \frac{1}{2}\sin(x+4x)+\frac{1}{2}\sin(x-4x)\,dx$$

$$= \frac{1}{2}\int \sin 5x\,dx + \frac{1}{2}\int \sin -3x\,dx$$

$$= -\frac{1}{10}\cos 5x + \frac{1}{6}\cos -3x + C$$

$$= \frac{1}{6}\cos 3x - \frac{1}{10}\cos 5x + C$$

④

$$\langle f, e_n\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x}e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-in)x}\,dx$$

$$= \frac{1}{2\pi}\,\frac{1}{1-in}\left[e^{(1-in)x}\right]_{-\pi}^{\pi} = \frac{1}{2\pi(1-in)}\left(e^{\pi}e^{-in\pi}-e^{-\pi}e^{in\pi}\right)$$

$$= \frac{(-1)^n}{\pi(1-in)}\left(\frac{e^{\pi}-e^{-\pi}}{2}\right) = \frac{(-1)^n\sinh\pi}{\pi(1-in)}$$ ← Not necessary to recognize.

So Fourier series of $e^x$ is:

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{\pi(1-in)}\sinh\pi\, e^{inx} = \frac{(-1)^0}{\pi}\sinh\pi + \left.\begin{array}{l}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\pi(1-in)}\sinh\pi\, e^{inx}\\ \displaystyle+\sum_{n=-1}^{-\infty}\frac{(-1)^n}{\pi(1-in)}\sinh\pi\, e^{inx}\end{array}\right\}\text{combine these}$$

$$= \frac{\sinh\pi}{\pi} + \frac{\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(1-in)}e^{inx} + \frac{(-1)^{-n}}{1+in}e^{-inx}$$

$$= \frac{\sinh\pi}{\pi} + \frac{\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\left(e^{inx}+e^{-inx}+ine^{inx}-ine^{-inx}\right)$$

$$= \frac{\sinh\pi}{\pi} + \frac{\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\left(2\cos nx + in\,2i\sin nx\right)$$

$$= \frac{\sinh\pi}{\pi} + \frac{\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\left(2\cos nx - 2n\sin nx\right)$$

(5)

$$\langle f, e_n \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 e^{-inx}\,dx$$

Integrate by parts.

$u = x^2 \quad v' = e^{-inx}$

$u' = 2x \quad v = \frac{1}{-in} e^{-inx}$

$$= \frac{1}{2\pi}\left[\frac{x^2}{-in} e^{-inx}\right]_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{2x}{-in} e^{-inx}\,dx$$

$$= \frac{\pi^2(-1)^n}{-2\pi in} - \frac{(-\pi)^2(-1)^n}{-2\pi in} + \frac{1}{in\pi}\int_{-\pi}^{\pi} x e^{-inx}\,dx$$

$$= \frac{1}{in\pi}\int_{-\pi}^{\pi} x\, e^{-inx}\,dx$$

Integrate by parts again

$u = x \quad v' = e^{-inx}$

$u' = 1 \quad v = \frac{1}{-in} e^{-inx}$

$$= \frac{1}{in\pi}\left(\left[\frac{x e^{-inx}}{-in}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{1}{-in} e^{-inx}\,dx\right)$$

$$= \frac{1}{in\pi}\left(\frac{\pi(-1)^n}{-in} - \frac{-\pi(-1)^n}{-in} + \frac{1}{in}\int_{-\pi}^{\pi} e^{-inx}\,dx\right)$$

$$= \frac{2\cancel{\pi}(-1)^n}{n^2\cancel{\pi}} - \frac{1}{n^2\pi}\left[\frac{1}{-in} e^{-inx}\right]_{-\pi}^{\pi}$$

$$= \frac{2}{n^2}(-1)^n - \frac{1}{n^2\pi}\left(\frac{(-1)^n}{-in} - \frac{(-1)^n}{-in}\right) = \frac{2}{n^2}(-1)^n$$

Note! None of this works if $n=0$, as we would then divide by zero.

We have $\langle f, e_0\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{2\pi}\left[\frac{1}{3}x^3\right]_{-\pi}^{\pi} = \frac{1}{\cancel{2\pi}}\frac{\cancel{2}\pi^{\cancel{3}}}{3} = \frac{\pi^2}{3}$

So the Fourier series is $\sum_{n=-\infty}^{\infty} a_n e^{inx} = a_0 + \sum_{n=1}^{\infty} a_n e^{inx} + \sum_{n=1}^{\infty} a_n e^{inx}$

$$\sum_{n=-\infty}^{\infty} a_n e^{inx} = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{2}{n^2}(-1)^n e^{inx} + \sum_{n=1}^{\infty}\frac{2}{(-n)^2}(-1)^{-n} e^{-inx}$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\left(e^{inx} + e^{-inx}\right) = \frac{\pi^3}{3} + \sum_{n=1}^{\infty}\frac{8}{n^2}(-1)^n \cos nx$$

⑥ We will rewrite $\sin\frac{x}{2}e^{-inx}$.

$$\sin\frac{x}{2}e^{-inx} = \sin\frac{x}{2}(\cos nx - i\sin nx) = \sin\frac{x}{2}\cos nx - i\sin\frac{x}{2}\sin nx$$

$$= \frac{1}{2}\sin\left(\frac{x}{2}+nx\right)+\frac{1}{2}\sin\left(\frac{x}{2}-nx\right) - i\left(\frac{1}{2}\cos\left(\frac{x}{2}-nx\right)-\frac{1}{2}\cos\left(\frac{1}{2}x+nx\right)\right)$$

by using ③ a) and d).

So we then get

$$\int_{-\pi}^{\pi}\sin\frac{x}{2}e^{-inx} = \frac{1}{2}\int_{-\pi}^{\pi}\sin\left(\frac{x}{2}+nx\right)dx+\frac{1}{2}\int_{-\pi}^{\pi}\sin\left(\frac{x}{2}-nx\right)dx - \frac{i}{2}\int_{-\pi}^{\pi}\cos\left(\frac{x}{2}-nx\right)dx+\frac{i}{2}\int_{-\pi}^{\pi}\cos\left(\frac{x}{2}+nx\right)dx$$

$$=\frac{1}{2}\left[\frac{-1}{\frac{1}{2}+n}\cos\left(\frac{x}{2}+nx\right)\right]_{-\pi}^{\pi}+\frac{1}{2}\left[\frac{-1}{\frac{1}{2}-n}\cos\left(\frac{x}{2}-nx\right)\right]_{-\pi}^{\pi}$$

$$-\frac{i}{2}\left[\frac{1}{\frac{1}{2}-n}\sin\left(\frac{x}{2}-nx\right)\right]_{-\pi}^{\pi}+\frac{i}{2}\left[\frac{1}{\frac{1}{2}+n}\sin\left(\frac{x}{2}+nx\right)\right]_{-\pi}^{\pi}$$

$$=\frac{1}{2}\frac{-1}{\frac{1}{2}+n}(0-0)+\frac{1}{2}\cdot\frac{-1}{\frac{1}{2}-n}(0-0)$$

$$-\frac{i}{2}\frac{1}{\frac{1}{2}-n}\left((-1)^n+(-1)^n\right)+\frac{i}{2}\frac{1}{\frac{1}{2}+n}\left((-1)^n+(-1)^n\right)$$

$$=(-1)^n i\left(\frac{1}{\frac{1}{2}+n}-\frac{1}{\frac{1}{2}-n}\right) = (-1)^n i\left(\frac{2}{1+2n}-\frac{2}{1-2n}\right)$$

Then the Fourier series looks like

$$\sum_{n=-\infty}^{\infty}(-1)^n i\left(\frac{2}{1+2n}-\frac{2}{1-2n}\right)e^{inx} = (-1)^0 i\left(\frac{2}{1+2\cdot 0}-\frac{2}{1-2\cdot 0}\right) + \sum_{n=1}^{\infty}(-1)^n i\left(\frac{2}{1+2n}-\frac{2}{1-2n}\right)e^{inx}$$

$$+\sum_{n=-1}^{-\infty}(-1)^n i\left(\frac{2}{1+2n}-\frac{2}{1-2n}\right)e^{inx}$$

$$=\sum_{n=1}^{\infty}(-1)^n i\left(\frac{2}{1+2n}-\frac{2}{1-2n}\right)e^{inx}+(-1)^{-n} i\left(\frac{2}{1-2n}-\frac{2}{1+2n}\right)e^{-inx}$$

$$=\sum_{n=1}^{\infty}(-1)^n i\left(\frac{2}{1+2n}-\frac{2}{1-2n}\right)\left(e^{inx}-e^{-inx}\right)$$

$$=\sum_{n=1}^{\infty}(-1)^n\left(\frac{2}{1+2n}-\frac{2}{1-2n}\right)i2i\sin nx$$

$$=\sum_{n=1}^{\infty}\left(\frac{4}{1+2n}-\frac{4}{1-2n}\right)(-1)^{n+1}\sin nx.$$

7)

a) Let us subtract $rS_n$ from $S_n$.

$$S_n = a_0 + a_0 r + a_0 r^2 + \dots + a_0 r^n$$
$$r S_n = \quad a_0 r + a_0 r^2 + \dots + a_0 r^n + a_0 r^{n+1}$$

So $S_n - rS_n = a_0 - a_0 r^{n+1}$

$$(1-r)S_n = a_0(1-r^{n+1})$$
$$S_n = \frac{a_0(1-r^{n+1})}{1-r}$$ as $r \neq 1$, we can divide by $1-r$.

This is what we wanted.

b) For $x \neq 2\pi n$, $e^{ikx} \neq 1$ so we can used the sum formula from a),

So $$\sum_{k=0}^{n} e^{ikx} = \frac{e^{i0x}(1-e^{i(n+1)x})}{1-e^{ix}} = \frac{1-e^{i(n+1)x}}{1-e^{ix}}$$

c) We manipulate the right hand side and try to get the same as in b).

$$e^{i\frac{nx}{2}} \frac{\sin\left(\frac{n+1}{2}x\right)}{\sin\frac{x}{2}} = e^{i\frac{nx}{2}} \frac{\frac{e^{i\frac{n+1}{2}x} - e^{-i\frac{n+1}{2}x}}{2i} \cdot 2ie^{i\frac{x}{2}}}{\frac{e^{i\frac{x}{2}} - e^{-i\frac{x}{2}}}{2i} \cdot 2ie^{i\frac{x}{2}}}$$

$$= e^{i\frac{nx}{2}} e^{i\frac{x}{2}} \frac{\left(e^{i\frac{n+1}{2}x} - e^{-i\frac{n+1}{2}x}\right)}{e^{ix} - 1}$$

$$= \frac{e^{i\frac{n+1}{2}x}\left(e^{i\frac{n+1}{2}x} - e^{-i\frac{n+1}{2}x}\right)}{e^{ix} - 1}$$

$$= \frac{e^{i(n+1)x} - 1}{e^{ix} - 1} = \frac{1 - e^{i(n+1)x}}{1 - e^{ix}} = \sum_{k=0}^{n} e^{ikx}$$

⑦ d) We have $\sum_{k=0}^{n} e^{ikx} = e^{i\frac{nx}{2}} \dfrac{\sin\frac{n+1}{2}x}{\sin\frac{x}{2}} = \left(\cos\frac{nx}{2} + i\sin\frac{nx}{2}\right)\dfrac{\sin\frac{n+1}{2}x}{\sin\frac{x}{2}}$

$$= \frac{\cos\frac{nx}{2}\sin\frac{n+1}{2}x}{\sin\frac{x}{2}} + i\,\frac{\sin\frac{nx}{2}\sin\frac{n+1}{2}x}{\sin\frac{x}{2}}$$

We also have $\sum_{k=0}^{n} e^{ikx} = \sum_{k=0}^{n} \cos kx + i\sin kx = \sum_{k=0}^{n} \cos kx + i\sum_{k=0}^{n} \sin kx$

By equating real and imaginary parts, we get

$$\sum_{k=0}^{n} \cos kx = \frac{\cos\frac{nx}{2}\sin\frac{n+1}{2}x}{\sin\frac{x}{2}} \quad \text{and}$$

$$\sum_{k=0}^{n} \sin kx = \frac{\sin\frac{nx}{2}\sin\frac{n+1}{2}x}{\sin\frac{x}{2}}$$ when $x$ is not a multiplom of $2\pi$.

If $x$ is a multiplom of $2\pi$, $\cos kx = 1$ and $\sin kx = 0$, so

$\sum_{k=0}^{n} \cos kx = n+1$ and $\sum_{k=0}^{n} \sin kx = 0$.

⑧ This exercise is the same as 5.3.7.
See the solution for that exercise.

7.21

① We want to show that $C_p$ is closed in $C([-\pi,\pi],\mathbb{C})$
Let us choose a sequence of functions $\{f_n\}$ in $C_p$ converging to an $f\in C([-\pi,\pi],\mathbb{C})$.
Then $f_n$ must converge pointwise to $f$, so $\lim_{n\to\infty} f_n(\pi)=f(\pi)$ and $\lim_{n\to\infty} f_n(-\pi)=f(-\pi)$,
But as $f_n\in C_p$, we have $f_n(\pi)=f_n(-\pi)$, so we get $f(\pi)=f(-\pi)$, and therefore $f\in C_p$. So every convergent sequence in $C_p$ converges to an element contained in $C_p \Rightarrow C_p$ closed. (NOTE: Only works with sup-norm, contrary to what I said during the plenary exercises.)

② a) As $f\in D$, there exists $a_1\leq\dots\leq a_n$, $a_1=-\pi$, $a_n=\pi$ s.t. $f$ is continuous on $(a_i,a_{i+1})$, $f(a_i^+)=\lim_{x\searrow a_i} f(x)$ and $f(a_i^-)=\lim_{x\nearrow a_i} f(x)$ exists and $f(a_i)=\dfrac{f(a_i^+)+f(a_i^-)}{2}$.

As $g\in D$, we can find $b_1,\dots,b_m$ s.t. the same attributes hold.

Choose $c_1,\dots,c_k$ s.t. $\{c_1,\dots,c_k\}=\{a_1,\dots,a_n\}\cup\{b_1,\dots,b_m\}$ and

$c_1<c_2<\dots<c_k$.

Then $f+g$ is continuous on each interval $(c_j,c_{j+1})$, as each such interval is contained both in an $(a_i,a_{i+1})$ and a $(b_\ell,b_{\ell+1})$, so both $f$ and $g$ are continuous. The same is true for $f\cdot g$, this is continuous as well.

As both $f(c_j^+)$ and $g(c_j^+)$ exists, we have $(f+g)(c_j^+)=f(c_j^+)+g(c_j^+)$, and $(f\cdot g)(c_j^+)=f(c_j^+)\cdot g(c_j^+)$, all these exist. The same is true for $c_j^-$

Lastly, $(f+g)(c_j)=f(c_j)+g(c_j)=\dfrac{f(c_j^+)+f(c_j^-)}{2}+\dfrac{g(c_j^+)+g(c_j^-)}{2}$
$=\dfrac{(f+g)(c_j^+)+(f+g)(c_j^-)}{2}$, as wanted.

---

However, we do <u>not</u> have $(f\cdot g)(c_j)=\dfrac{(f\cdot g)(c_j^+)+(f\cdot g)(c_j^-)}{2}$ in general, so for $f\cdot g$ to be in $D$, we have to redefine the values at $c_j$. Which we can do, but the exercise as written is wrong.

② b)

If $f, g \in D$, we have seen that $f+g \in D$, and $\alpha f$ will also be in $D$.
Must therefore check the definition of a vector space.

i) $(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x)$, so $f+g = g+f$.

ii) $((f+g)+h)(x) = (f+g)(x) + h(x) = (f(x) + g(x)) + h(x)$
$= f(x) + (g(x) + h(x)) = f(x) + (g+h)(x)$
$= (f+(g+h))(x)$, so $(f+g)+h = f+(g+h)$.

iii) Choose $\bar{0}(x) = 0$, the function that is constantly zero.
Then $\bar{0} \in C_p \subseteq D$, so $\bar{0} \in D$. We have $(f+\bar{0})(x) = f(x) + \bar{0}(x) = f(x) + 0 = f(x)$,
so $f + \bar{0} = f$.

iv) Given $f \in D$, let $(-f)(x) = -(f(x))$. Then $-f = -1 \cdot f \in D$, and we have
$(f+(-f))(x) = f(x) + (-f)(x) = f(x) - f(x) = 0 = \bar{0}(x)$, so $f + -f = \bar{0}$.

v) $(\alpha(f+g))(x) = \alpha(f+g)(x) = \alpha(f(x) + g(x)) = \alpha f(x) + \alpha g(x)$
$= (\alpha f)(x) + (\alpha g)(x) = (\alpha f + \alpha g)(x)$, so $\alpha(f+g) = \alpha f + \alpha g$.

vi) $((\alpha+\beta)f)(x) = (\alpha+\beta)f(x) = \alpha f(x) + \beta f(x) = (\alpha f)(x) + (\beta f)(x)$
$= (\alpha f + \beta f)(x)$, so $(\alpha+\beta)f = \alpha f + \beta f$.

vii) $(\alpha(\beta f))(x) = \alpha(\beta f)(x) = \alpha(\beta f(x)) = (\alpha\beta)f(x) = ((\alpha\beta)f)(x)$ so $\alpha(\beta f) = (\alpha\beta)f$.

viii) $(1 \cdot f)(x) = 1 \cdot f(x) = f(x)$, so $1f = f$.

c) If $f \in D$ we can on each interval $(a_i, a_{i+1})$ look at the function

$$f_i(x) = \begin{cases} f(x) & x \in (a_i, a_{i+1}) \\ f(a_i^-) & x = a_i \\ f(a_{i+1}^+) & x = a_{i+1} \end{cases}$$

Then $f_i(x)$ is continuous on the compact interval $[a_i, a_{i+1}]$, and must therefore have a maximum $M_i$.

Let $M = \max_i \{M_i\}$, $N = \max\{f(a_i)\}$, $K = \max(M, N)$.

Then $f(x) \leq K$ for all $x \in [-\pi, \pi]$, so $f$ is bounded.

d) For each interval $(a_i, a_{i+1})$ we have that $\int_{a_i}^{a_{i+1}} f(x)dx = \int_{a_i}^{a_{i+1}} f_i(x)dx$

where $f_i$ is defined as in c).

These integrals are equal as the functions only differ in two points, $a_i$ and $a_{i+1}$.

The integrals $\int^{a_{i+1}} f_i(x)dx$ do exist as they are integrals over continuous functions.

Then $\int_{-\pi}^{\pi} f(x)dx = \int_{a_1}^{a_2} f(x)dx + \int_{a_2}^{a_3} f(x)dx + \dots + \int_{a_{n-1}}^{a_n} f(x)dx$

$= \int_{a}^{a_2} f_1(x)dx + \dots + \int_{a_{n-1}}^{a_n} f_{n-1}(x)dx$ exists, so $f$ is integrable.

e) Most of what we did in 7.11 (1) works here as well, so just check that exercise.

The only thing to note is that in 7.11 (1) we had that

$\int |f(x)|^2 dx = 0 \Rightarrow f(x) \equiv 0$ because $f(x)$ was continuous.

Here we instead get that $f(x) \equiv 0$ on each interval $(a_i, a_{i+1})$,

but then $f(a_i^+) = 0$, $f(a_i^-) = 0$, so $f(a_i) = \frac{0+0}{2} = 0$, and $f \equiv 0$ on $[-\pi, \pi]$.

③

a) $\alpha_{-n} = \langle f, e_{-n} \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-i(-n)x}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, \overline{e^{-inx}}\,dx$

$= \overline{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\,dx} = \overline{\alpha_n}$ as wanted

b) We have $a_n = \frac{\alpha_n + \overline{\alpha_n}}{2} = \frac{1}{2}(\alpha_n + \alpha_{-n})$

$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \left(\frac{e^{-inx} + e^{inx}}{2}\right) dx$

$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \cos x\, dx$ as wanted,

and $b_n = \frac{\alpha_n - \overline{\alpha_n}}{2i} = \frac{1}{2i}(\alpha_n - \alpha_{-n})$

$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, \frac{e^{-inx} - e^{inx}}{2i}\, dx$

$= \frac{-1}{2\pi}\int_{-\pi}^{\pi} f(x)\, \frac{e^{inx} - e^{-inx}}{2i}\, dx$

$= -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$ as wanted.

c) The Fourier series of $f$ is given by

$$\sum_{n=-\infty}^{\infty} \alpha_n e_n(x) = \alpha_0 e_0(x) + \sum_{n=1}^{\infty} \alpha_n e_n(x) + \sum_{n=-\infty}^{-1} \alpha_n e_n(x)$$

$$= \alpha_0 e^{i0x} + \sum_{n=1}^{\infty} \alpha_n e^{inx} + \sum_{n=1}^{\infty} \alpha_{-n} e^{-inx}$$

$$= \alpha_0 + \sum_{n=1}^{\infty} a_n e^{inx} + i b_n e^{inx} + a_n e^{-inx} - i b_n e^{-inx}$$

$$= \alpha_0 + \sum_{n=1}^{\infty} a_n (e^{inx} + e^{-inx}) + i b_n (e^{inx} - e^{-inx})$$

$$= \alpha_0 + \sum_{n=1}^{\infty} a_n 2 \cos nx + i b_n 2i \sin nx$$

$$= \alpha_0 + \sum_{n=1}^{\infty} 2a_n \cos nx - 2 b_n \sin nx \qquad \square$$