7.21

(4) As g is a line between two points of f, there will always exist an x' s.t. $|f(x)-g(x)| \le |f(x)-f(x')|$

So $\sup_{x\in(-\delta+a_i,\delta+a_i)} |f(x)-g(x)| \le \sup_{x\in(-\delta+a_i,\delta+a_i)} |f(x)-f(x')| \le \sup_{x\in(-\delta+a_i,\delta+a_i)} \{|f(x)|+|f(x')|\}$

$$\le \sup_{x\in(-\delta+a_i,\delta+a_i)} |f(x)| + \sup_{x'\in(-\delta+a_i,\delta+a_i)} |f(x')|$$

$$\le M+M = 2M.$$

Therefore, we have $\int_{-\delta+a_i}^{\delta+a_i} |f(x)-g(x)|^2 dx \le 2\delta \sup_{x\in(-\delta+a_i,\delta+a_i)} |f(x)-g(x)|^2 \le 2\delta\cdot 4M^2 = 8\delta M^2$

This is true for any δ and any a_i

We want $\|f-g\|_2^2 < \varepsilon^2$

$$\int_{-\pi}^{\pi} |f(x)-g(x)|^2 dx = \int_{-\pi}^{-\pi+\delta} |f(x)-g(x)|^2 dx + \int_{-\pi+\delta}^{-\delta+a_1} |f(x)-g(x)|^2 dx + \int_{-\delta+a_1}^{\delta+a_1} |f(x)-g(x)|^2 dx$$

$$+ \int_{a_1+\delta}^{-\delta+a_2} |f(x)-g(x)|^2 dx + \int_{-\delta+a_2}^{\delta+a_2} |f(x)-g(x)|^2 dx$$

$$\vdots$$

$$+ \int_{\delta+a_{n-1}}^{\pi-\delta} |f(x)-g(x)|^2 dx + \int_{\pi-\delta}^{\pi} |f(x)-g(x)|^2 dx$$

$$\le \int_{-\pi}^{-\pi+\delta} |f(x)-g(x)|^2 dx + (n-1)\cdot 0 + (n-2)\cdot 8\delta M^2 + \int_{\pi-\delta}^{\pi} |f(x)-g(x)|^2 dx$$

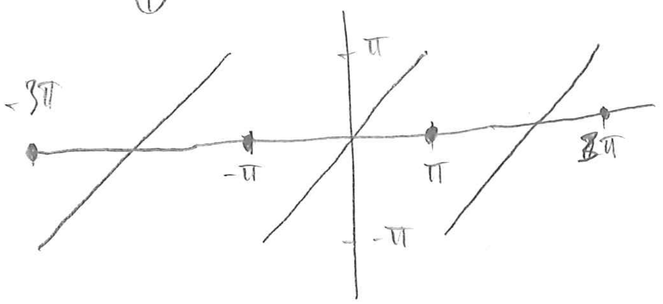
By periodicity, $\int_{-\pi}^{-\pi+\delta} |f(x)-g(x)|^2 dx + \int_{\pi-\delta}^{\pi} |f(x)-g(x)|^2 dx = \int_{\pi-\delta}^{\pi+\delta} |f(x)-g(x)|^2 dx \le 8\delta M^2$,

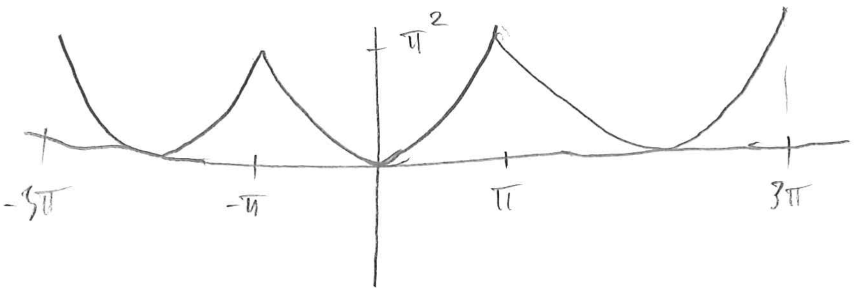
So $\int_{-\pi}^{\pi} |f(x)-g(x)|^2 dx \le (n-1)\, 8\delta M^2$

We want $(n-1)\,8\delta M^2 < \varepsilon^2$, so we choose any $\delta < \frac{\varepsilon^2}{(n-1)8M^2}$,

This proves the lemma.

7.31 ①





②
$$\lim_{t\to 0} \frac{\sin\left(\left(n+\frac{1}{2}\right)t\right)}{\sin \frac{t}{2}} \overset{\text{L'Hop}}{=} \lim_{t\to 0} \frac{\left(n+\frac{1}{2}\right)\cos\left(\left(n+\frac{1}{2}\right)t\right)}{\frac{1}{2}\cos\frac{t}{2}} = \frac{n+\frac{1}{2}}{\frac{1}{2}} = 2n+1$$

③ We have

$u = x - t$, $t = x - u$

$\frac{du}{dt} = -1$

$-du = dt$

$$S_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\sum_{n=-N}^{N} e^{in(x-t)}\,dt$$

$$= \frac{1}{2\pi}\int_{x+\pi}^{x-\pi} -f(x-u)\sum_{n=-N}^{N} e^{inu}\,du$$

$$= \frac{1}{2\pi}\int_{-\pi+x}^{\pi+x} f(x-u)\sum_{n=-N}^{N} e^{inu}\,du$$

$v = u - 2\pi$

$\frac{dv}{du} = 1$

$dv = du$

$$= \frac{1}{2\pi}\int_{-\pi+x}^{\pi} f(x-u)\sum_{n=-N}^{N} e^{inu}\,du + \frac{1}{2\pi}\int_{\pi}^{\pi+x} f(x-u)\sum_{n=-N}^{N} e^{inu}\,du$$

$$= \frac{1}{2\pi}\int_{-\pi+x}^{\pi} f(x-u)\sum_{n=-N}^{N} e^{inu}\,du + \frac{1}{2\pi}\int_{-\pi}^{-\pi+x} f(x-v-2\pi)\sum_{n=-N}^{N} e^{inv}e^{in2\pi}\,dv$$

$$= \frac{1}{2\pi}\int_{-\pi+x}^{\pi} f(x-u)\sum_{n=-N}^{N} e^{inu}\,du + \frac{1}{2\pi}\int_{-\pi}^{-\pi+x} f(x-v)\sum_{n=-N}^{N} e^{inv}\,dv = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)\sum_{n=-N}^{N} e^{inu}\,du$$

④ As $|\sin u| \leq |u|$, we have $\displaystyle |D_n(t)| = \frac{|\sin((n+\frac{1}{2})t)|}{|\sin\frac{t}{2}|} \geq \frac{|\sin((n+\frac{1}{2})t)|}{|\frac{t}{2}|}$

$$= \frac{2|\sin((n+\frac{1}{2})t)|}{|t|}.$$

So $\displaystyle\int_{-\pi}^{\pi} |D_n(t)|\,dt = \int_{-\pi}^{0} |D_n(t)|\,dt + \int_{0}^{\pi} |D_n(t)|\,dt$

$t' = -t$

$\frac{dt'}{dt} = -1$

$dt' = -dt$

$$= -\int_{\pi}^{0} |D_n(-t')|\,dt' + \int_{0}^{\pi} |D_n(t)|\,dt$$

$$= \int_{0}^{\pi} |D_n(t')|\,dt' + \int_{0}^{\pi} |D_n(t)|\,dt$$

$$= \int_{0}^{\pi} 2|D_n(t)|\,dt \geq \int_{0}^{\pi} 2\,\frac{2|\sin((n+\frac{1}{2})t)|}{|t|}\,dt$$

$z = (n+\frac{1}{2})t$

$\frac{dz}{dt} = n+\frac{1}{2}$

$$= \int_{0}^{(n+\frac{1}{2})\pi} \frac{4|\sin z|}{(n+\frac{1}{2})|t|}\,dz$$

as $z > 0$ $\quad \frac{dz}{(n+\frac{1}{2})} = dt$

$$= \int_{0}^{(n+\frac{1}{2})\pi} \frac{4|\sin z|}{z}\,dz$$

$$= \int_{0}^{\pi} \frac{4|\sin z|}{z}\,dz + \int_{\pi}^{2\pi} \frac{4|\sin z|}{z}\,dz + \int_{2\pi}^{3\pi} \frac{4|\sin z|}{z} + \dots + \int_{(n-1)\pi}^{n\pi} \frac{4|\sin z|}{z}\,dz + \int_{n\pi}^{(n+\frac{1}{2})\pi} \frac{4|\sin z|}{z}\,dz$$

$$= \sum_{k=1}^{n}\left(\int_{(k-1)\pi}^{k\pi} \frac{4|\sin z|}{z}\,dz\right) + \int_{n\pi}^{(n+\frac{1}{2})\pi} \frac{4|\sin z|}{z}\,dz \geq \sum_{k=1}^{n} \int_{(k-1)\pi}^{k\pi} \frac{4|\sin z|}{z}\,dz$$

as $z \leq k\pi$ on each interval

$$\geq \sum_{k=1}^{n} \int_{(k-1)\pi}^{k\pi} \frac{4|\sin z|}{k\pi}\,dz = \sum_{k=1}^{n} \frac{4}{k\pi} \int_{(k-1)\pi}^{k\pi} |\sin z|\,dz = \sum_{k=1}^{n} \frac{4}{k\pi}\cdot 2 = \frac{8}{\pi}\sum_{k=1}^{n}\frac{1}{k}.$$

7.4

(1) $C\text{-}\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} a_k}{n}$

We have that $\sum_{k=0}^{n-1} a_k = \frac{n}{2}$ if n is even

and $\sum_{k=0}^{n-1} a_k = \frac{n-1}{2}$ if n is odd

So we have $\frac{\sum_{k=0}^{n-1} a_k}{n} = \frac{1}{2}$ if n is even and

$\frac{\sum_{k=0}^{n-1} a_k}{n} = \frac{n-1}{2n}$ if n is odd.

As $\lim_{n\to\infty} \frac{n-1}{2n} = \frac{1}{2}$ we can find an N s.t. $\left|\frac{n-1}{2n} - \frac{1}{2}\right| < \varepsilon$ for all $n \geq N$, and therefore also for all odd $n \geq N$.

So $\left|\frac{\sum_{k=0}^{n-1} a_k}{n} - \frac{1}{2}\right| < \varepsilon$ for all $n \geq N$, both odd and even,

which proves $C\text{-}\lim a_n = \frac{1}{2}$

②

$$C\text{-}\lim_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty}\frac{\sum_{k=0}^{n-1} a_k+b_k}{n} = \lim_{n\to\infty}\frac{\sum_{k=0}^{n-1} a_k + \sum_{k=0}^{n-1} b_k}{n}$$

$$= \lim_{n\to\infty}\frac{\sum_{k=0}^{n-1} a_k}{n} + \lim_{n\to\infty}\frac{\sum_{k=0}^{n-1} b_k}{n}$$

$$= C\text{-}\lim_{n\to\infty} a_n + C\text{-}\lim_{n\to\infty} b_n$$

③ $$\lim_{u\to 0}\frac{\sin^2\left(\frac{nu}{2}\right)}{n\sin^2\left(\frac{u}{2}\right)} \overset{\text{L'Hop}}{=} \lim_{u\to 0}\frac{2\sin\left(\frac{nu}{2}\right)\cdot\frac{n}{2}\cos\left(\frac{nu}{2}\right)}{2n\sin\frac{u}{2}\cdot\frac{1}{2}\cos\frac{u}{2}}$$

$$= \lim_{u\to 0}\frac{\cancel{\frac{n}{2}}\sin(nu)}{\cancel{\frac{n}{2}}\sin(u)} = \lim_{u\to 0}\frac{\sin nu}{\sin u}$$

$$\overset{\text{L'Hop}}{=} \lim_{u\to 0}\frac{n\cos(nu)}{\cos u} = \frac{n\cdot 1}{1} = n$$

④ $$S_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)F_N(u)\,du = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)\frac{1}{N}\sum_{k=0}^{N-1} D_k(u)\,du$$

$$= \frac{1}{2\pi N}\sum_{k=0}^{N-1}\int_{-\pi}^{\pi} f(x-u)\sum_{n=-k}^{k} e^{iun}\,du$$

$t = x-u$
$u = x-t$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{n=-k}^{k}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt\; e^{inx}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{n=-k}^{k} a_n e^{inx}$$

In this double sum, each $a_n e^{inx}$ is counted once for every loop where $k \geq |n|$, i.e. $N-|n|$ times.

By $\longrightarrow$

$$= \frac{1}{N}\sum_{n=-(N-1)}^{N-1} a_n(N-|n|)e^{inx} = \sum_{n=-(N-1)}^{N-1} a_n\left(1-\frac{|n|}{N}\right)e^{inx}$$ as wanted.

(5)

Given $f \in C_p$, show S_n converges to f uniformly.

Given $\varepsilon > 0$, find $N \in \mathbb{N}$ s.t. $|S_n(x) - f(x)| < \varepsilon$ for all x when $n \geq N$,

As f is continuous on a compact, f is uniformly continuous, and we can find a $\delta > 0$ s.t. $|f(x+u) + f(x-u) - 2f(x)| < \varepsilon$ when $|u| < \delta$.
The same δ works for all x.

We have $|S_n(x) - f(x)| \leq \frac{1}{4\pi}\int_{-\pi}^{\pi} |f(x+u) + f(x-u) - 2f(x)| F_n(x) du$ $\quad |\int \cdot\, du| \leq \int |\cdot|\, du$

$$= \frac{1}{4\pi}\int_{-\delta}^{\delta} |f(x+u) + f(x-u) - 2f(x)| F_n(x) du \quad (1)$$

$$+ \frac{1}{4\pi}\int_{-\pi}^{-\delta} |f(x+u) + f(x-u) - 2f(x)| F_n(x) du \quad (2)$$

$$+ \frac{1}{4\pi}\int_{\delta}^{\pi} |f(x+u) + f(x-u) - 2f(x)| F_n(x) du \quad (3)$$

For (1): $\frac{1}{4\pi}\int_{-\delta}^{\delta} |f(x+u) + f(x-u) - 2f(x)| F_n(u) du \leq \frac{1}{4\pi}\int_{-\delta}^{\delta} \varepsilon F_n(x) du$

$$\leq \frac{\varepsilon}{4\pi}\int_{-\pi}^{\pi} F_n(u) du = \frac{\varepsilon}{2}\frac{1}{2\pi}\int_{-\pi}^{\pi} F_n(u) du = \frac{\varepsilon}{2}$$

For (2),(3): $\frac{1}{4\pi}\int_{-\pi}^{-\delta} |f(x+u) + f(x-u) - 2f(x)| F_n(u) du$

$$\leq \frac{1}{4\pi}\int_{-\pi}^{-\delta} (|f(x+u)| + |f(x-u)| + 2|f(x)|) F_n(u) du$$

$$\leq \frac{1}{4\pi}\int_{-\pi}^{-\delta} (\|f\|_\infty + \|f\|_\infty + 2\|f\|_\infty) F_n(u) du$$

$$= \frac{\|f\|_\infty}{\pi}\int_{-\pi}^{-\delta} F_n(u) du \leq \frac{\|f\|_\infty}{\pi}\int_{-\pi}^{-\delta} \frac{\pi^2}{n\delta^2} du \leq \frac{\pi^2 \|f\|_\infty}{n\delta^2}$$

Same for (3). Choose $N \geq \frac{4\pi^2 \|f\|_\infty}{\varepsilon \delta^2}$ to get (2) $< \frac{\varepsilon}{4}$, (3) $< \frac{\varepsilon}{4}$,

so $|S_n(x) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon$ for all $x \in [-\pi, \pi]$, 7.5|

① $\int_a^b f(x)\cos(nx)dx = \int_a^b f(x)\frac{e^{inx}+e^{-inx}}{2}dx = \frac{1}{2}\int_a^b f(x)e^{inx}dx + \frac{1}{2}\int_a^b f(x)e^{-inx}dx$

$= \frac{1}{2}\int_{-b}^{-a} f(-t)e^{-int}dt + \frac{1}{2}\int_a^b f(x)e^{-inx}dx$

$= \pi a_n' + \pi a_n \to 0$

where a_n is the Fourier coefficient of $g(x)$ and a_n' is the Fourier coefficient of the corresponding function for $f(-x)$

$\int_a^b f(x)\sin nx\,dx = \int_a^b f(x)\frac{e^{inx}-e^{-inx}}{2i}dx = \frac{1}{2i}\int_a^b f(x)e^{inx} - \frac{1}{2i}\int_a^b f(x)e^{-inx}dx$

$= \frac{1}{2i}\int_{-b}^{-a} f(-t)e^{-int}dt - \frac{1}{2i}\int_a^b f(x)e^{-inx}dx$

$= \frac{\pi}{i}a_n' - \frac{\pi}{i}a_n \to 0$

where a_n is the Fourier coefficients of $g(x)$ and a_n' is the Fourier coefficients of the corresponding function for $f(-x)$.

② $\int_a^b f(x)\sin((n+\frac{1}{2})x)dx = \int_a^b f(x)\sin nx\cos\frac{x}{2} + f(x)\cos nx\sin\frac{x}{2}\,dx$

$= \int_a^b f(x)\sin nx\cos\frac{x}{2}dx + \int_a^b f(x)\cos nx\sin\frac{x}{2}dx$

$= \int_a^b g(x)\sin nx\,dx + \int_a^b h(x)\cos nx\,dx$ where $g(x) = f(x)\cos\frac{x}{2}$, $h(x) = f(x)\sin\frac{x}{2}$

Then $g(x)\in D$, $h(x)\in D$, so by Corr 7.3.3,

$\int_a^b f(x)\sin((n+\frac{1}{2})x)dx = \int_a^b g(x)\sin nx\,dx + \int_a^b h(x)\cos nx\,dx$

$\to 0 + 0 = 0$ when $|n|\to\infty$

③ a) p is a trig. polynomial $\Leftrightarrow$ $p(x) = \sum_{k=-N}^{N} a_k e^{ikx}$,

$\langle p, e_n\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} p(x)e^{-inx}dx = \sum_{k=-N}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi} a_k e^{ikx}e^{-inx}dx = \sum_{k=-N}^{N}\frac{a_k}{2\pi}\int_{-\pi}^{\pi} e^{i(k-n)x}dx$

$\int_{-\pi}^{\pi} e^{i(k-n)x}dx = 0$ if $k\neq n$, so choose $n\geq N$. Then $\langle p, e_n\rangle = 0$.

b) Choose an $\varepsilon > 0$, and a $p(t)$ s.t. $\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t)-p(t)|dt < \varepsilon$.

Then, for large enough $|n|$ $a_n = \langle f, e_n\rangle = \langle f, e_n\rangle - \langle p, e_n\rangle$

as $\langle p, e_n\rangle = 0$. so $|a_n| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}(f(t)-p(t))e^{-int}dt\right| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t)-p(t)||e^{-int}|dt$

$= \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t)-p(t)|dt < \varepsilon$.

Therefore, $\lim_{|n|\to\infty} a_n = 0$.