7.61

① Should be example 7.1.1.

We define $g(x) = \begin{cases} x & x \in (-\pi, \pi) \\ 0 & x = \pm\pi \end{cases}$

Then $g \in D$, and for any point $x \in (-\pi, \pi)$ we have that $g$ is differentiable at $x$. So the Fourier series of $g$ must converge pointwise to $g(x) = x$. As $f(x)$ and $g(x)$ have the same Fourier series (their integrals will be equal, as they only differ in two pts), we have that $\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx) = x$ for all $x \in (-\pi, \pi)$.

For the endpoints, we have that

$$\lim_{u \downarrow 0} \frac{g(-\pi+u) - g(-\pi^+)}{u} = \lim_{u \downarrow 0} \frac{-\pi + u - (-\pi)}{u} = \lim_{u \downarrow 0} \frac{u}{u} = 1$$

and $$\lim_{u \uparrow 0} \frac{g(\pi+u) - g(\pi^+)}{u} = \lim_{u \uparrow 0} \frac{\pi + u - \pi}{u} = \lim_{u \uparrow 0} \frac{u}{u} = 1,$$

both exist, so $$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(n\pi) = g(\pi) = 0$$

and $$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(-n\pi) = g(-\pi) = 0$$

Indeed, $\sin(n\pi) = 0$, so these are just infinite sums of zeros.

②

If $f$ is differentiable at $x$, the limit

$$\lim_{u \to 0} \frac{f(x+u)-f(x)}{u}$$

exists. So we can choose a $\delta$ so small that

$$\left| \frac{f(x+u)-f(x)}{u} \right| < |f'(x)| + \varepsilon$$

for all $u < \delta$.

Then
$$\left| \frac{f(x+u)+f(x-u)-2f(x)}{u} \right| \leq \left| \frac{f(x+u)-f(x)}{u} \right| + \left| \frac{f(x-u)-f(x)}{u} \right|$$
$$< 2|f'(x)| + 2\varepsilon$$

whenever $u \in (-\delta, \delta)$.

So
$$\int_{-\delta}^{\delta} \left| \frac{f(x+u)+f(x-u)-2f(x)}{u} \right| du \leq \int_{-\delta}^{\delta} 2|f'(x)| + 2\varepsilon \, du$$
$$= (2|f'(x)| + 2\varepsilon) 2\delta < \infty,$$

and by Dini's test, the ~~Fourier~~ series converges at $x$.

③ We have $f(x) = \frac{f(x^+) + f(x^-)}{2}$, so $2f(x) = f(x^+) + f(x^-)$.

As both limits from Cor 7.6.4 exists, we can find a $\delta > 0$ s.t. $\left|\frac{f(x+u) - f(x^+)}{u}\right| < |M_+| + \varepsilon$ and $\left|\frac{f(x-u) - f(x^-)}{u}\right| < |M_-| + \varepsilon$

for all $0 < u < \delta$, where $M_+$ and $M_-$ are the limits themselves.

Then we get $\int_{-\delta}^{\delta} \left|\frac{f(x+u) + f(x-u) - 2f(x)}{u}\right| du$

$$= \int_{-\delta}^{0} \left|\frac{f(x+u) + f(x-u) - 2f(x)}{u}\right| du + \int_{0}^{\delta} \left|\frac{f(x+u) + f(x-u) - 2f(x)}{u}\right| du$$

$$= -\int_{\delta}^{0} \left|\frac{f(x-v) + f(x+v) - 2f(x)}{-v}\right| dv + \int_{0}^{\delta} \left|\frac{f(x+u) + f(x-u) - 2f(x)}{u}\right| du$$

$$= 2\int_{0}^{\delta} \left|\frac{f(x+u) + f(x-u) - 2f(x)}{u}\right| du$$

$$= 2\int_{0}^{\delta} \left|\frac{f(x+u) + f(x-u) - f(x^+) - f(x^-)}{u}\right| du$$

$$\leq 2\int_{0}^{\delta} \left|\frac{f(x+u) - f(x^+)}{u}\right| + \left|\frac{f(x-u) - f(x^-)}{u}\right| du$$

$$\leq 2\int_{0}^{\delta} |M_+| + \varepsilon + |M_-| + \varepsilon \, du$$

$$= 2\delta(|M_+| + |M_-| + 2\varepsilon) < \infty,$$

So by Dini's test, the ~~Fourier~~ series converges in $x$.

4

a) We have that $f(x)$ is continuous and differentiable on $[-\pi, \pi]$ (check this), so the Fourier-series will converge to $f$, i.e.,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \text{where} \quad c_n = \langle f, e_n \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx.$$

Let us work with $c_n$.

$$\int_{-\pi}^{\pi} \frac{\sin x}{x} e^{-inx}\,dx = \int_{-\pi}^{\pi} \frac{e^{ix}-e^{-ix}}{2ix} e^{-inx}\,dx$$

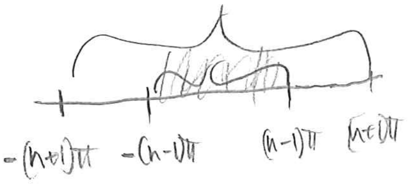
$$= \int_{-\pi}^{\pi} \frac{e^{-ix(n-1)}}{2ix} - \frac{e^{-ix(n+1)}}{2ix}\,dx$$

$$= \int_{-\pi}^{\pi} \frac{e^{-ix(n-1)}}{2ix}\,dx - \int_{-\pi}^{\pi} \frac{e^{-ix(n+1)}}{2ix}\,dx$$

$z = x(n-1)$ in first integral

$z = x(n+1)$ in second integral.

$$= \int_{-\pi(n-1)}^{\pi(n-1)} \frac{e^{-iz}}{2iz}\,dz - \int_{-\pi(n+1)}^{\pi(n+1)} \frac{e^{-iz}}{2iz}\,dz$$




$$= -\left( \int_{-(n+1)\pi}^{(n+1)\pi} \frac{e^{-iz}}{2iz}\,dz - \int_{-(n-1)\pi}^{(n-1)\pi} \frac{e^{-iz}}{2iz}\,dz \right)$$

$$= -\left( \int_{-(n+1)\pi}^{-(n-1)\pi} \frac{e^{-iz}}{2iz}\,dz + \int_{(n-1)\pi}^{(n+1)\pi} \frac{e^{-iz}}{2iz}\,dz \right)$$

$z \mapsto -z$ in first integral,

$$= -\left( \int_{(n+1)\pi}^{(n-1)\pi} \frac{e^{iz}}{2iz}\,dz + \int_{(n-1)\pi}^{(n+1)\pi} \frac{e^{-iz}}{2iz}\,dz \right)$$

$$= -\int_{(n+1)\pi}^{(n-1)\pi} \frac{e^{iz}}{2iz}\,dz - \int_{(n-1)\pi}^{(n+1)\pi} \frac{e^{-iz}}{2iz}\,dz$$

$$= \int_{(n-1)\pi}^{(n+1)\pi} \frac{e^{iz}}{2iz} - \frac{e^{-iz}}{2iz}\,dz = \int_{(n-1)\pi}^{(n+1)\pi} \frac{\sin z}{z}\,dz$$ as wanted.

(v) We note $f(0) = \sum_{n=-\infty}^{\infty} c_n e^{in0} = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{(n-1)\pi}^{(n+1)\pi} \frac{\sin z}{z} dz$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \left( \int_{(n-1)\pi}^{n\pi} \frac{\sin z}{z} dz + \int_{n\pi}^{(n+1)\pi} \frac{\sin z}{z} dz \right)$$

$$= \frac{1}{2\pi} \left( \sum_{n=-\infty}^{\infty} \int_{(n-1)\pi}^{n\pi} \frac{\sin z}{z} dz + \sum_{n=-\infty}^{\infty} \int_{n\pi}^{(n+1)\pi} \frac{\sin z}{z} dz \right)$$

$$= \frac{1}{2\pi} \left( \int_{-\infty}^{\infty} \frac{\sin z}{z} dz + \int_{-\infty}^{\infty} \frac{\sin z}{z} dz \right)$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin z}{z} dz$$

and $f(0) = 1$. So $\int_{-\infty}^{\infty} \frac{\sin z}{z} dz = \pi$

5

a) We have $\sum_{n=-\infty}^{\infty} r^{|n|} e^{inx} = 1 + \sum_{n=1}^{\infty} r^{n} e^{inx} + \sum_{n=-1}^{\infty} r^{-n} e^{inx}$

$$= 1 + \sum_{n=1}^{\infty} r^{n} e^{inx} + \sum_{n=1}^{\infty} r^{n} e^{-inx}$$

We have $|r^n e^{inx}| = r^n$ and $|r^n e^{-inx}| = r^n$, so by Weierstrass M-test, this converges uniformly.

To find the sum we rewrite the sum as

$$\sum_{n=-\infty}^{\infty} r^{|n|} e^{inx} = 1 + \sum_{n=1}^{\infty} (re^{ix})^n + \sum_{n=1}^{\infty} (re^{-ix})^n$$

$$= 1 + \frac{re^{ix}}{1-re^{ix}} + \frac{re^{-ix}}{1-re^{-ix}}$$

$$= \frac{(1-re^{ix})(1-re^{-ix})}{(1-re^{ix})(1-re^{-ix})} + \frac{re^{ix}(1-re^{-ix})}{(1-re^{ix})(1-re^{-ix})} + \frac{re^{-ix}(1-re^{ix})}{(1-re^{ix})(1-re^{-ix})}$$

$$= \frac{1 \cancel{-re^{-ix}} \cancel{-re^{ix}} \cancel{+r^2} \cancel{+re^{ix}} \cancel{-r^2} \cancel{+re^{-ix}} - r^2}{1-re^{-ix}-re^{ix}+r^2}$$

$$= \frac{1-r^2}{1-r(e^{ix}+e^{-ix})+r^2} = \frac{1-r^2}{1-2r\cos x+r^2}$$

b) We have $1-r^2 > 0$, so we need to show that $1-2r\cos(x)+r^2 > 0$ as well. The smallest this can possibly be is when $\cos x = 1$, and then we get $1-2r+r^2 = (1-r)^2 > 0$ as $1-r \neq 0$.

c) When $|x| \geq \delta$, we have $\cos(x) < \cos\delta$, so

$$P_r(x) = \frac{1-r^2}{1-2r\cos x + r^2} \leq \frac{1-r^2}{1-2r\cos\delta + r^2}$$

$$\lim_{r \nearrow 1} \frac{1-r^2}{1-2r\cos\delta + r^2} = \frac{0}{2-2\cos\delta} = 0 \quad \text{as } \cos\delta \neq 1.$$

So as $P_r(x) \leq \frac{1-r^2}{1-2r\cos\delta+r^2} \to 0$ as $r \nearrow 1$, we have uniform convergence on $[-\pi, -\delta] \cup [\delta, \pi]$.

d)
$$\int_{-\pi}^{\pi} P_r(x)\,dx = \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} r^{|n|} e^{inx}\,dx = \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} r^{|n|} e^{inx}\,dx$$

$$= \sum_{n=-\infty}^{\infty} r^{|n|} \int_{-\pi}^{\pi} e^{inx}\,dx = 2\pi$$

as all these integrals are $0$ except for $n=0$.

We can exchange the sum and the integral as we have uniform convergence (a).

e)
$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-y) P_r(y)\,dy - f(x) \right| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x-y) - f(x)| P_r(y)\,dy$$

$$= \frac{1}{2\pi}\left( \int_{-\pi}^{-\delta} |f(x-y) - f(x)| P_r(y)\,dy + \int_{-\delta}^{\delta} |f(x-y)-f(x)| P_r(y)\,dy + \int_{\delta}^{\pi} |f(x-y) - f(x)| P_r(y)\,dy \right)$$

Can choose $\delta$ so small that $|f(x-y) - f(x)| < \frac{\varepsilon}{2}$ whenever $x \in (-\delta, \delta)$

$$\leq \frac{1}{2\pi} \int_{-\pi}^{-\delta} |f(x-y) - f(x)| P_r(y)\,dy + \frac{1}{2\pi} \int_{\delta}^{\pi} |f(x-y) - f(x)| P_r(y)\,dy + \frac{\varepsilon}{2} \frac{1}{2\pi} \int_{-\delta}^{\delta} P_r(y)\,dy$$

e) cont.

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)P_r(y)\,dy - f(x)\right| < \frac{1}{2\pi}\int_{[-\pi,-\delta]\cup[\delta,\pi]} |f(x-y)-f(x)|P_r(y)\,dy + \frac{\varepsilon}{2}\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(y)\,dy$$

$$= \frac{1}{2\pi}\int_{[-\pi,-\delta]\cup[\delta,\pi]} |f(x-y)-f(x)|P_r(y)\,dy + \frac{\varepsilon}{2}$$

Can choose $r$ close to $1$ s.t. $P_r(y) < \frac{\varepsilon}{4\cdot\|f\|_\infty}$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \cancel{2\|f\|_\infty}\frac{\varepsilon}{\cancel{4\|f\|_\infty}}\,dy + \frac{\varepsilon}{2}$$

$$= \varepsilon, \quad \text{as wanted.}$$

f) Let $g(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)P_r(y)\,dy$ $\quad u = x-y$, $du = dy$

$$= \frac{1}{2\pi}\int_{x+\pi}^{x-\pi} -f(u)\,P_r(u+x)\,du$$

By $2\pi$-periodicity

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)\,P_r(u+x)\,du$$

So $g'(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)\,P_r'(u+x)\,du$, and as $P_r(u+x)$ is differentiable, $g(x)$ is also differentiable. By Dini's test, the Fourier series of $g(x)$ will converge to $g(x)$, and we have

$$a_n = \langle g, e_n\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x-y)P_r(y)e^{-inx}\,dy\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x-y)e^{-inx}\,dx\,P_r(y)\,dy \qquad u = x-y,\ du = dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi-y}^{\pi-y} f(u)\,e^{-inu}\,du\,e^{-iny}P_r(y)\,dy$$

f) cont.

$$\alpha_n = \int_{-\pi}^{\pi} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)e^{-inu}du \; P_r(y)e^{-iny}dy$$

$$= \int_{-\pi}^{\pi} c_n P_r(y)e^{-iny}dy = c_n \int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty} r^{|k|}e^{iky}e^{-iny}dy$$

$$= c_n \sum_{k=-\infty}^{\infty} r^{|k|}\int_{-\pi}^{\pi} e^{i(k-n)y}dy = c_n r^{|n|}.$$

So $g(x) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in} = \sum_{n=\infty}^{\infty} c_n r^{|n|}e^{in}$ as wanted.

g)

We now have $\lim_{r\nearrow 1}\sum_{n=-\infty}^{\infty} c_n r^{|n|}e^{inx} \overset{f)}{=} \lim_{r\nearrow 1}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)P_r(y)dy \overset{e)}{=} f(x).$

7.7.

① Let $\beta_n$ be the Fourier coefficients of $g(x) - a_0 x$.

Remember that $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$ and $g'(x) = f(x)$.

We first compute $\beta_n$ for $n \neq 0$.

$$\beta_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}(g(x) - a_0 x)e^{-inx}dx$$

$u = g(x) - a_0 x \quad v = -\frac{1}{in}e^{-inx}$

$u' = f(x) - a_0 \quad v' = e^{-inx}$

$$= \frac{1}{2\pi}\left[-\frac{1}{in}e^{-inx}(g(x) - a_0 x)\right]_{-\pi}^{\pi} + \frac{1}{2\pi i n}\int_{-\pi}^{\pi} f(x)e^{-inx} - a_0 e^{-inx} dx$$

$$= \frac{1}{2\pi}\left(-\frac{1}{in}\left((-1)^n(g(\pi) - a_0\pi) - (-1)^n(g(-\pi) + a_0\pi)\right)\right) + \frac{1}{in}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

$$- \frac{1}{2\pi i n}\int a_0 e^{-inx}dx$$

$$= \frac{-1\cdot(-1)^n}{2\pi i n}\left(g(\pi) - g(-\pi) - 2\pi a_0\right) + \frac{1}{in}a_n - 0$$

$$= \frac{(-1)^{n+1}}{2\pi i n}\cancel{\left(\int_0^{\pi} f(x)dx - \int_0^{-\pi} f(x)dx - \int_{-\pi}^{\pi} f(x)dx\right)} + \frac{a_n}{in} = \frac{a_n}{in} = -\frac{i a_n}{n}$$

For $n=0$ we have: $g(0) - a_0 \cdot 0 = \int_0^0 f(x)dx - 0 = 0$

$$= \beta_0 + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \beta_n e^{i0} = \beta_0 + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \beta_n$$

So $\beta_0 = -\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \beta_n = \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{i a_n}{n}$.

Then the Fourier series of $g(x) - a_0 x$ is given by

$$\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{i a_n}{n} - \sum_{\substack{n\in\mathbb{Z}\\ \cancel{n\neq 0}}} \frac{i a_n}{n} e^{inx}$$ as wanted.

(2)

This exercise has nothing directly to do with Fourier Series, but will be used in the next exercise.

$$\sum_{k=1}^{n} \cos(2k-1)x = \frac{\sin 2nx}{2\sin x}$$

$$\sum_{k=1}^{n} \cos\big((2k-1)x\big) = \sum_{k=1}^{n} \frac{e^{i(2k-1)x} + e^{-i(2k-1)x}}{2} = \frac{1}{2}\left(\underbrace{\sum_{k=1}^{n} e^{i(2k-1)x} + \sum_{k=1}^{n} e^{-i(2k-1)x}}\right)$$

$$= \frac{1}{2}\left(e^{ix}\,\frac{e^{i2nx}-1}{e^{i2x}-1} + e^{-ix}\,\frac{e^{-i2nx}-1}{e^{-i2x}-1}\right)$$ Geometric Series

$$= \frac{1}{2}\left(\frac{e^{i2nx}-1}{e^{ix}-e^{-ix}} + \frac{e^{-i2nx}-1}{e^{-ix}-e^{ix}}\right)$$

$$= \frac{1}{2}\left(\frac{e^{i2nx}-1-(e^{-i2nx}-1)}{e^{ix}-e^{-ix}}\right)$$

$$= \frac{1}{\cancel{2}}\left(\frac{\cancel{2i}\sin 2nx}{\cancel{2i}\sin x}\right) = \frac{\sin 2nx}{\sin x}$$ as wanted.

③

a) Fourier coefficients of $f$:

$$2\pi a_n = \int_{-\pi}^{\pi} f(x) e^{-inx} dx = \int_{-\pi}^{0} f(x) e^{-inx} dx + \int_{0}^{\pi} f(x) e^{-inx} dx$$

$n=0$:

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{0} -1 \, dx + \int_{0}^{\pi} 1 \, dx = 0$$

$$= -\int_{-\pi}^{0} e^{-inx} dx + \int_{0}^{\pi} e^{-inx} dx$$

$$= +\int_{\pi}^{0} e^{inx} dx + \int_{0}^{\pi} e^{-inx} dx$$

$$= \int_{0}^{\pi} e^{-inx} - e^{inx} dx = \int_{0}^{\pi} 2i \sin(-nx) dx$$

$$= -\int_{0}^{\pi} 2i \sin nx \, dx = \frac{2i}{n}\Big[\cos nx\Big]_{0}^{\pi} = \frac{2i((-1)^n - 1)}{n}$$

so 0 if $n$ even, $\frac{-4i}{n}$ if $n$ odd

$$S_{2n-1}(x) = \sum_{\substack{N=-(2n-1) \\ N \text{ odd}}}^{2n-1} a_N e^{iNx} = \sum_{\substack{N=-(2n-1) \\ N \text{ odd}}}^{2n-1} \frac{-2i}{\pi N} e^{iNx}$$

$$= \sum_{\substack{N=1 \\ N \text{ odd}}}^{2n-1} \frac{-2i}{\pi N} e^{iNx} + \sum_{\substack{N=-1 \\ N \text{ odd}}}^{-(2n-1)} \frac{-2i}{\pi N} e^{iNx}$$

$N=2k-1$ always odd

$$= \sum_{\substack{N=1 \\ N \text{ odd}}}^{2n-1} \frac{-2i}{\pi N} \left(e^{iNx} - e^{-iNx}\right) = \sum_{\substack{N=1 \\ N \text{ odd}}}^{2n-1} \frac{4}{\pi N} \sin Nx$$

$$= \sum_{k=1}^{n} \frac{4}{(2k-1)\pi} \sin((2k-1)x) = \frac{4}{\pi} \sum_{k=1}^{n} \frac{\sin((2k-1)x)}{2k-1}.$$

b) We have $\left(\sum_{k=1}^{n} \frac{\sin(2k-1)x}{2k-1}\right)' = \sum_{k=1}^{n} \cos(2k-1)x = \frac{\sin 2nx}{2\sin x}$

So $S'_{2n-1}(x) = \frac{4}{\pi}\frac{\sin 2nx}{2\sin x} = \frac{2}{\pi}\frac{\sin 2nx}{\sin x}$

c) Local minima/maxima when $S'_{2n-1}(x) = 0$, i.e. $\sin 2nx = 0$
i.e. $2nx = k\cdot\pi$.
$k=0, \Rightarrow x=0$ is <u>not</u> a solution, as then $2\sin x = 0$ as well, and we get $\frac{0}{0} \to \frac{4n}{\pi}$.
So solutions closest to zero are $2nx = \pm\pi$, i.e. $x = \pm\frac{\pi}{2n}$.

d) $S_{2n-1}\left(\pm\frac{\pi}{2n}\right) = \frac{4}{\pi}\sum_{k=1}^{n}\frac{\sin \pm\frac{(2k-1)\pi}{2n}}{2k-1} = \pm\frac{4}{\pi}\sum_{k=1}^{n}\frac{\sin\left(\frac{(2k-1)\pi}{2n}\right)}{2k-1}$

e) Let $\Delta x = \frac{\pi}{n}$. Then we have

$$\pm\frac{4}{\pi}\sum_{k=1}^{n}\frac{\sin\frac{(2k-1)\pi}{2n}}{2k-1} = \pm\frac{4}{\pi}\sum_{k=1}^{n}\frac{\sin((k+\frac{1}{2})\Delta x)}{2(k+\frac{1}{2})\Delta x}\Delta x$$

$$= \pm\frac{2}{\pi}\sum_{k=1}^{n} f((k+\tfrac{1}{2})\Delta x)\Delta x \quad \text{where } f(x) = \frac{\sin x}{x}.$$

This is a (middle) Riemann-sum, so will converge to $\pm\frac{2}{\pi}\int_0^{\pi}\frac{\sin x}{x}\,dx$.

f) I used WolframAlpha, but anything should work.

5.4

1. Induction over $n$.

$\underline{n=1:}$ $A(\alpha_1 u_1) = \alpha_1 A(u_1)$ By (i).

Assume OK for $n=k$, show for $n=k+1$.

$$A(\alpha_1 v_1 + \dots + \alpha_k v_k + \alpha_{k+1} v_{k+1}) = A(\alpha_1 v_1 + \dots + \alpha_k v_k) + A(\alpha_{k+1} v_{k+1}) \quad \text{By (ii)}$$
$$= \alpha_1 A(v_1) + \dots + \alpha_k A(v_k) + A(\alpha_{k+1} v_{k+1}) \quad \text{By I.H.}$$
$$= \alpha_1 A(v_1) + \dots + \alpha_k A(v_k) + \alpha_{k+1} A(v_{k+1}) \quad \text{By (i).}$$

2. We have $A(\alpha u) = \int_a^b \alpha u(x)\,dx = \alpha \int_a^b u(x)\,dx = \alpha A(u)$

and $A(u+v) = \int_a^b u(x)+v(x)\,dx = \int_a^b u(x)\,dx + \int_a^b v(x)\,dx = A(u) + A(v)$.

So $A$ linear.

For $B$ we have that $B(\alpha u)(x) = \int_a^x \alpha u(t)\,dt = \alpha \int_a^x u(t)\,dt = \alpha B(u)(x)$

so $B(\alpha u) = B(u)$

and $B(u+v)(x) = \int_a^x u(t)+v(t)\,dt = \int_a^x u(t)\,dt + \int_a^x v(t)\,dt = B(u)(x) + B(v)(x)$
$= (B(u)+B(v))(x).$

So $B(u+v) = B(u) + B(v)$

3. $D(\alpha u)(x) = (\alpha u)'(x) = \alpha u'(x) = \alpha D(u)(x)$, so $D(\alpha u) = \alpha D(u)$

$D(u+v)(x) = (u+v)'(x) = u'(x) + v'(x) = D(u)(x) + D(v)(x) = (D(u)+D(v))(x)$

so $D(u+v) = D(u) + D(v)$

4

Let $u_n$ be a sequence of nonzero vectors s.t. $\lim_{n\to\infty} \frac{\|A(u_n)\|_W}{\|u_n\|_V} = \|A\|$.

This must exist, as $\|A\|$ is the supremum of such values.

Let $v_n = \frac{u_n}{\|u_n\|_V}$. Then $\|v_n\| = \left\|\frac{u_n}{\|u_n\|_V}\right\|_V = \frac{\|u_n\|_V}{\|u_n\|_V} = 1$, and we have

$$\|A(v_n)\|_W = \left\|A\left(\frac{u_n}{\|u_n\|_V}\right)\right\|_W = \left\|\frac{A(u_n)}{\|u_n\|_V}\right\|_W = \frac{\|A(u_n)\|_W}{\|u_n\|_V} \to \|A\| \text{ as } n\to\infty.$$

As each $\|v_n\| = 1$, we must have $\sup\{\|A(v)\|_W : \|v\|_V = 1\} \geq \sup\{\frac{\|A(u)\|_W}{\|u\|_V} : u \neq 0\}$.

We also have that $\sup\{\|A(v)\|_W : \|v\|_V = 1\} \leq \sup\{\frac{\|A(u)\|_W}{\|u\|_V} : u \neq 0\}$,

as $\|A(v)\|_W = \frac{\|A(v)\|_W}{\|v\|_V}$, so one set is contained in the other.

Combine these and we get equality.

5

$$\left.\begin{aligned} F(u+v) &= (u+v)(0) = u(0)+v(0) = F(u)+F(v) \\ F(\alpha u) &= (\alpha u)(0) = \alpha u(0) = \alpha F(u) \end{aligned}\right\} F \text{ linear.}$$

We have that $\|u\|_\infty = \sup_{x\in[0,1]}\{|u(x)|\} \geq |u(0)| = \|F(u)\|$, so $F$ is bounded.

$F$ is therefore also continuous, by Thm 5.4.5.

6

We have $C(u+v) = B(A(u+v)) = B(A(u)+A(v)) = B(A(u)) + B(A(v)) = C(u)+C(v)$.

and $C(\alpha u) = B(A(\alpha u)) = B(\alpha A(u)) = \alpha B(A(u)) = \alpha C(u)$.

so $C$ is linear.

For any $u$ we have $\|A\|\|u\|_U \geq \|A(u)\|_V$ and $\|B\|\|A(u)\|_V \geq \|B(A(u))\|_W$, so

we have $\|C(u)\|_W \leq \|B\|\|A(u)\|_V \leq \|B\|\|A\|\|u\|_U$, so $C$ is Bounded.

We then see that $\|C\| \leq \|B\|\|A\|$ as well.

If we let $U = \mathbb{R}^2$, $V = \mathbb{R}^2$, $W = \mathbb{R}^2$, let $A(x,y) = (x,0)$, $B(x,y) = (0,y)$, we get

$C(x,y) = (0,0)$, so $\|C\| = 0$. And $\|A\| = \|B\| = 1$.

⑦

Check all the requirements:

i) $(A+B)(u) = A(u)+B(u) = B(u)+A(u) = (B+A)(u)$ as $W$ is a vector space.

ii) $((A+B)+C)(u) = (A+B)(u)+C(u) = (A(u)+B(u))+C(u) = A(u)+(B(u)+C(u))$

$= A(u)+(B+C)(u) = (A+(B+C))(u)$

iii) There is a zero operator $O(u)=\bar{0}$, $(A+O)(u) = A(u)+O(u) = A(u)+\bar{0} = A(u)$.

iv) For each $A$, define $-A$ by $(-A)(u) = -(A(u))$.

$(-A)(u+v) = -(A(u+v)) = -(A(u)+A(v)) = -(A(u)) + -(A(v))$

$= (-A)(u) + (-A)(v)$

$(-A)(\alpha u) = -(A(\alpha u)) = -(\alpha A(u)) = \alpha \cdot -A(u) = \alpha(-A)(u)$

So $-A \in \mathcal{L}(V,W)$.

v) $(\alpha(A+B))(u) = \alpha((A+B)(u)) = \alpha(A(u)+B(u)) = \alpha A(u)+\alpha B(u) = (\alpha A)(u)+(\alpha B)(u)$

$= (\alpha A+\alpha B)(u)$.

vi) $((\alpha+\beta)A)(u) = (\alpha+\beta)A(u) = \alpha A(u)+\beta A(u) = (\alpha A)(u)+(\beta A)(u) = (\alpha A+\beta A)(u)$

vii) $(\alpha(\beta A))(u) = \alpha(\beta A)(u) = \alpha(\beta(A(u))) = (\alpha\beta)A(u) = ((\alpha\beta)A)(u)$

viii) $(1A)(u) = 1\cdot A(u) = A(u)$