5.41

(8) Let $x \in \mathbb{R}^d$, $x = (x_1, \dots, x_d) = x_1\bar{e}_1 + \dots + x_d\bar{e}_d$.

Want to show $\|A(x)\|_W \leq K\|x\|$ for some K, independent of x.

Have:

$$\|A(x)\|_W^2 = \|A(x_1\bar{e}_1 + \dots + x_d\bar{e}_d)\|_W^2 = \|x_1A(e_1) + \dots + x_dA(e_d)\|_W^2$$

$$\leq (|x_1|\|A(e_1)\|_W + \dots + |x_d|\|A(e_d)\|_W)^2 \leq x_1^2\|A(e_1)\|_W^2 + \dots + x_d^2\|A(e_d)\|_W^2$$

$$\leq x_1^2K^2 + \dots + x_d^2K^2 = K^2(x_1^2 + \dots + x_d^2) = K^2\|x\|^2$$

where $K = \max_{i \leq d} \|A(e_i)\|_W$. This implies $\|A(x)\|_W \leq K\|x\|$ for all x.

(10) a) $B(\alpha\vec{u} + \beta\vec{v}) = \langle \alpha\vec{u} + \beta\vec{v}, \vec{y} \rangle = \alpha\langle \vec{u}, \vec{y} \rangle + \beta\langle \vec{v}, \vec{y} \rangle = \alpha B(\vec{u}) + \beta B(\vec{v})$

so B is linear.

$\|B(u)\|_{\mathbb{R}} = |B(u)| = |\langle \vec{u}, \vec{y} \rangle| \leq \|\vec{u}\|_V \cdot \|\vec{y}\|_V$ for all $\vec{u}$, so B is bounded.

b) We have $A(\sum_{i=1}^n \beta_i e_i) = \sum_{i=1}^n A(\beta_i e_i) = \sum_{i=1}^n \beta_i A(e_i) = \sum_{i=1}^n \beta_i^2$

As A is bounded, we have $\|A(\sum_{i=1}^n \beta_i e_i)\|_{\mathbb{R}}^2 \leq \|A\|^2 \cdot \|\sum_{i=1}^n \beta_i e_i\|_V^2$

So

$$\left|\sum_{i=1}^n \beta_i^2\right|^2 \leq \|A\|^2 \|\sum \beta_i e_i\|_V^2 = \left|\sum_{i=1}^n \beta_i^2\right|^2 \leq \|A\|^2 \left(\sum_{i=1}^n \beta_i^2\right)$$

by Pythagoras. So $\sum_{i=1}^n \beta_i^2 \leq \|A\|^2 \Rightarrow \left(\sum_{i=1}^n \beta_i^2\right)^{1/2} \leq \|A\|$.

This is true for all $n \in \mathbb{N}$, so will also be true for the limit.

c) By Prop. 5.3.11, $\sum_{i=1}^\infty \beta_i e_i$ will converge, as $\{\beta_i\}$ are square summable, by b).

d) Let $x = \sum_{i=1}^{\infty} \alpha_i e_i$, $x = \lim_{n\to\infty} \sum_{i=1}^{n} \alpha_i e_i$; with $\langle x, e_i \rangle = \alpha_i$

As A is bounded, A must be continuous, so $A(x) = A(\lim_{n\to\infty} \sum_{i=1}^{n} \alpha_i e_i) = \lim_{n\to\infty} A(\sum_{i=1}^{n} \alpha_i e_i)$

$$A(x) = \lim_{n\to\infty} A(\sum_{i=1}^{n} \alpha_i e_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \alpha_i A(e_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \alpha_i \beta_i = \sum_{i=1}^{\infty} \alpha_i \beta_i.$$

We have $\langle x, y \rangle = \langle x, \sum_{i=1}^{\infty} \beta_i e_i \rangle$

$$= \sum_{i=1}^{\infty} \langle x, \beta_i e_i \rangle = \sum_{i=1}^{\infty} \beta_i \langle x, e_i \rangle = \sum_{i=1}^{\infty} \beta_i \alpha_i = \sum_{i=1}^{\infty} \alpha_i \beta_i = A(x)$$

by Prop 5.3.7. (iv) and Parsevals theorem.

From a) we saw that $\|A\| \leq \|y\|_V$.

But $A(y) = \langle y, y \rangle = \|y\|_V^2 = \|y\|_V \cdot \|y\|_V$, so $\|A\| \geq \|y_V\|$ as well.

Therefore $\|A\| = \|y\|_V$.

5.5.

①

Choose any open set $G \subseteq \mathbb{R}$, and any ball $B(x,\varepsilon)$ with $B(x,\varepsilon) \subseteq G$ and $\varepsilon < 1$. As $\varepsilon < 1$, $B(x,\varepsilon)$ can contain at most one point from $\mathbb{N}$. If there is no such point, i.e. $B(x,\varepsilon) \cap \mathbb{N} = \emptyset$, we are done. Otherwise, let $n \in \mathbb{N} \cap B(x,\varepsilon)$ and choose $y \in B(x,\varepsilon)$, $y \neq n$. Let $\delta = \min\left\{\frac{|y-n|}{2}, \frac{\varepsilon - |y-x|}{2}\right\}$.

Then $B(y,\delta) \subseteq B(x,\varepsilon)$ and $n \notin B(y,\delta)$. As n was the only natural number in $B(x,\varepsilon)$, we have $B(y,\delta) \cap \mathbb{N} = \emptyset$, as wanted.

② Choose any open set $G \subseteq C([0,1],\mathbb{R})$, and any ball $B(f,\varepsilon) = \{h : \sup_{x\in[0,1]}\{|f(x)-h(x)|\} < \varepsilon\} \subseteq G$. If $f(x) \neq 0$, then let $\delta = \min\{\varepsilon, |f(0)|\}$. Then $B(f,\delta) \subseteq B(f,\varepsilon) \subseteq G$ and for any $g \in A$, $\sup_{x\in[0,1]}\{|f(x)-g(x)|\} \geq |f(0)-g(0)| = |f(0)| \geq \delta$, so $g \notin B(f,\delta)$, i.e. $B(f,\delta) \cap A = \emptyset$.

If $f(0)=0$, we can let $h(x) = \frac{\varepsilon}{2}$. Then $h \in B(f,\varepsilon)$, we can find an open ball $B(h,\varepsilon') \subseteq B(f,\varepsilon) \subseteq G$, $h(0) \neq 0$. By the same argument as above, we find δ s.t. $B(h,\delta) \subseteq G$, $B(h,\delta) \cap A = \emptyset$, as wanted.

③ Let $A \subseteq B$, B nowhere dense in X. Then for each open $G \subseteq X$, there exists an open ball D in G s.t. $D \cap B = \emptyset$. But as $A \subseteq B$, we have $D \cap A = \emptyset$ as well, so A is nowhere dense.

Let $A \subseteq B$, B meager. Then we can write $B = \bigcup_{i\in I} B_i$, where each B_i is nowhere dense, and I is countable.

Let $A_i = A \cap B_i$. As $A_i \subseteq B_i$, we have that A_i is nowhere dense.

We have $\bigcup_{i\in I} A_i = \bigcup_{i\in I} A \cap B_i = A \cap \left(\bigcup_{i\in I} B_i\right) = A \cap B = A$, so A is meager.

④

If S is nowhere dense, then for each open $G \subseteq X$, there exists an open ball $B(x;r)$ that does not intersect S. This is then also true when G is of the form $B(a_0;r_0)$.

If, for each open ball $B(a_0;r_0)$ there exists a ball $B(x;r) \subseteq B(a_0;r_0)$, not intersecting S, then for any open $G \subseteq X$ we can find a ball $B(a_0,r_0) \subseteq G$, $B(x,r) \subseteq B(a_0,r_0)$, $B(x,r) \cap S = \emptyset$. So we have an open ball in G not intersecting S for all open sets G, and S is nowhere dense.

⑤

a) As N is nowhere dense, for any $G \subseteq X$ open, we can find a ball $B(x,r) \subseteq G$, $B(x,r) \cap N = \emptyset$. Assume $B(x,r) \cap \bar{N} \neq \emptyset$, and let $y \in B(x,r) \cap \bar{N}$. As $B(x,r) \cap N = \emptyset$, $y \notin N$, so y must be a boundary point, $B(y,\varepsilon)$ contains points in N and in N^c.
But as $N \subseteq B(x,r)^c$ and $y \in B(x,r)$, we have that $B(y,\varepsilon)$ contains points both in $B(x,r)$ and in $B(x,r)^c$, i.e. y is a boundary point for $B(x,r)$. However, $B(x,r)$ is open and hence contains no boundary points, a contradiction. So we must have $B(x,r) \cap \bar{N} = \emptyset$, so $\bar{N}$ is nowhere dense.

b) $\mathbb{Q}$ is meager, but $\bar{\mathbb{Q}} = \mathbb{R}$ is not.

c) If N is nowhere dense, $\bar{N}$ is nowhere dense and hence does not contain any open balls by Lemma 5.5.5.

If $\bar{N}$ does not contain any open balls, $\bar{N}$ is nowhere dense, and then N is nowhere dense by $N \subseteq \bar{N}$ and ③.

⑥ Let $A = \bigcup_{i \in I} A_i$, where each A_i is meager, and I is countable.

As A_i is meager, $A_i = \bigcup_{j \in J_i} B_j$ where B_j is nowhere dense and J_i is countable.

Let $K = \{(i,j) \mid i \in I, j \in J_i\}$. Then K is countable.

Let $C_k = B_j$ where $k = (i,j)$, $j \in J_i$.

Then $A = \bigcup_{i \in I} \bigcup_{j \in J_i} B_j = \bigcup_{k \in K} C_k$ is a countable union of nowhere dense sets, so A is meager.

⑦ Let G be any open set. As N_1 is nowhere dense, there exists an open ball $B(a_1, r_1) \subseteq G$ s.t. $B(a_1, r_1) \cap N_1 = \emptyset$.

As N_2 is nowhere dense, there exists an open ball $B(a_2, r_2) \subseteq B(a_1, r_1)$ s.t. $B(a_2, r_2) \cap N_2 = \emptyset$.

As N_3 is nowhere dense, there exists an open ball $B(a_3, r_3) \subseteq B(a_2, r_2)$ s.t. $B(a_3, r_3) \cap N_3 = \emptyset$.

Repeat, find ball $B(a_i, r_i) \subseteq B(a_{i-1}, r_{i-1})$ with $B(a_i, r_i) \cap N_i = \emptyset$.

Then $B(a_k, r_k) \subseteq B(a_{k-1}, r_{k-1}) \subseteq \dots \subseteq B(a_1, r_1)$, so

$$B(a_k, r_k) \cap N_k = \emptyset,\ B(a_k, r_k) \cap N_{k-1} \subseteq B(a_{k-1}, r_{k-1}) \cap N_{k-1} = \emptyset,$$
$$\dots\ B(a_k, r_k) \cap N_1 \subseteq B(a_1, r_1) \cap N_1 = \emptyset$$

so $B(a_k, r_k) \subseteq G$ and $B(a_k, r_k) \cap \left(\bigcup_{i=1}^{k} N_i\right) = \emptyset$.

This works for all G, so $N_1 \cup \dots \cup N_k$ is nowhere dense.

⑧

If S is nowhere dense, $\overline{S}$ is nowhere dense, and does not contain any open balls. Then $\overline{S}^c$ is open and contained in S^c, and for any $x \in X$, $B(x,\varepsilon)$ must contain points in $\overline{S}^c$, otherwise we would have $B(x,\varepsilon) \subseteq \overline{S}$. So $\overline{S}^c$ is dense.

If S is not nowhere dense, $\overline{S}$ must contain at least one open ball $B(x,\varepsilon)$, and then no sequence in $\overline{S}^c$ can converge onto x, i.e. $\overline{S}^c$ is not dense. And then there cannot exist any open dense set in S^c, as it would be contained in $\overline{S}^c$.

⑨ a) Should be 5.5.9.

$$g_n(x) = \begin{cases} nx & x < \frac{\varepsilon}{2n} \\ \frac{\varepsilon}{2} & x \geq \frac{\varepsilon}{2n} \end{cases}$$

$\{g_n\}$ equicontinuous $\Leftrightarrow$ For all $\varepsilon' > 0$ there exists a $\delta > 0$ s.t. for all $n \in \mathbb{N}$, for all $x, y \in [0,1]$: $|x-y| < \delta \Rightarrow |g_n(x) - g_n(y)| < \varepsilon'$.

So $\{g_n\}$ not equicontinuous $\Leftrightarrow$ There exists $\varepsilon' > 0$ s.t. for all $\delta > 0$ there exists $n \in \mathbb{N}$, $x, y \in [0,1]$ with $|x-y| < \delta$, but $|g_n(x) - g_n(y)| \geq \varepsilon'$.

Let $\varepsilon' = \frac{\varepsilon}{4}$ (as in the def. of g_n). Then, given any $\delta > 0$ we can choose $x=0$, $y = \frac{\delta}{2}$, n s.t. $\frac{\varepsilon}{2n} < \frac{\delta}{2}$.

Then $|x-y| = |\frac{\delta}{2}| = \frac{\delta}{2} < \delta$, but $|g_n(x) - g_n(y)| = \frac{\varepsilon}{2} > \frac{\varepsilon}{4}$, so $\{g_n\}$ is not equicontinuous.

(9)

b) Want to show that given an $\varepsilon > 0$ we can find a $\delta > 0$ s.t. for all $n \in \mathbb{N}$, $x, y \in [0,1]$, $|h_n(x) + k(x) - h_n(y) - k(y)| < \varepsilon$ whenever $|x-y| < \delta$.

As $\{h_n\}$ is equicontinuous, we can find a $\delta_1 > 0$ s.t. for all $n \in \mathbb{N}$, $x, y \in [0,1]$ we have $|h_n(x) - h_n(y)| < \frac{\varepsilon}{2}$ whenever $|x-y| < \delta_1$.

As k is continuous on a compact, k is uniformly continuous, so we can find a $\delta_2 > 0$ s.t. for all $x, y \in [0,1]$ we have $|k(x) - k(y)| < \frac{\varepsilon}{2}$ whenever $|x-y| < \delta_2$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Then we have

$$|h_n(x) + k(x) - h_n(y) - k(y)| \leq |h_n(x) - h_n(y)| + |k(x) - k(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $|x-y| < \delta$, so $\{h_n + k\}$ is equicontinuous.

c) Assume $\{f + g_n\}$ equicontinuous. Let $h_n = f + g_n$ and $k = -f$. Then k is continuous, so by b) $\{h_n + k\} = \{f + g_n - f\} = \{g_n\}$ should be equicontinuous. But by a) $\{g_n\}$ is not equicontinuous, so $\{f + g_n\}$ cannot be either.

(10)

$\mathbb{N}$ is complete as $\{x_k\}$ Cauchy $\Rightarrow$ $x_k = a$ for all $k \geq N$ for some $N \in \mathbb{N}$

$\Rightarrow \lim_{k \to \infty} x_k = a$.

$\mathbb{N} = \bigcup_{n \in \mathbb{N}} \{n\}$ as for each $k \in \mathbb{N}$ we have $k \in \{k\} \subseteq \bigcup_{n \in \mathbb{N}} \{n\}$.

Does not contradict Baire's Category theorem, as $\{n\}$ is not nowhere dense. The set $\{n\}$ is open, and it is dense in itself (for all sets A, A is dense in A).

Therefore $\mathbb{N}$ is not meager.

(11)
If a set is nowhere dense, it must be meager, as it's the countable union of itself.

If a set is meager, it does not contain any open balls, and as it's closed it must be nowhere dense by Lemma 5.5.5.

(12)
a) We have that $(G^c)^c$ contains an open, dense subset (G itself). So by (8) G^c is nowhere dense.

b) We have $\left(\bigcap_{n\in\mathbb{N}} G_n\right)^c = \bigcup_{n\in\mathbb{N}} G_n^c$. By a) G_n^c is nowhere dense, so

$\bigcup_{n\in\mathbb{N}} G_n^c$ is a countable union of nowhere dense sets, i.e. meager.
By Baire's category theorem, $\left(\bigcap_{n\in\mathbb{N}} G_n\right)^c$ does not contain any open balls, so for any $x \in X$, $\varepsilon > 0$, $B(x,\varepsilon)$ must contain points in $\bigcap_{n\in\mathbb{N}} G_n$, i.e.

$\bigcap_{n\in\mathbb{N}} G_n$ is dense.

(13) As $f_n(x) \to f(x)$, $\{f_n\}$ must be pointwise bounded (all convergent sequences are bounded), so by Prop 5.5.7 there exists an open $G \subseteq [0,1]$ s.t. $\{f_n\}$ is bounded on G.
Choose any point $x \in G$, there must exist ε s.t. $(x-\varepsilon, x+\varepsilon) \subseteq G$, and then we have that $\{f_n\}$ is bounded on this subinterval, and then f is bounded as well

Let $f_n(x) = \begin{cases} \frac{1}{x} & x \geq \frac{1}{n} \\ n^2 x & x < \frac{1}{n} \end{cases}$. Then $f_n(x)$ is continuous, and converges

pointwise to $f(x) = \begin{cases} \frac{1}{x} & x > 0 \\ 0 & x = 0 \end{cases}$, but f is not bounded on $[0,1]$.

(14)

a) As each f_n is continuous, $f_n^{-1}([-1,1])$ is closed.

Let us define $A_k = \bigcap_{n \geq k} f_n^{-1}([-1,1])$. Then A_k is closed, and as $\lim_{n\to\infty} f_n(x) = 0$, we have $\bigcup_k A_k = [0,1]$. As $[0,1]$ is complete, it cannot be meager, so at least one A_k must be not nowhere dense.

Assume A_N is not nowhere dense. As A_N is closed, it must contain at least one "open ball" (a,b). Then, for each $x \in (a,b)$, $x \in A_N \Rightarrow x \in f_n^{-1}([-1,1])$ for all $n \geq N$, i.e. $|f_n(x)| \leq 1$ for all $n \geq N$, as we wanted.

b) (There must be a nicer example than this, but i could not think of any).

Let p_n be the nth prime number, and let

$$f_n(x) \begin{cases} 2 & \text{if } x = \frac{k}{p_n^l} \text{ for some } k, l \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Then for each $x \in [0,1]$, $f_n(x) = 0$ for all but possibly one n, so $f_n(x) \to 0$ for all x. But if (a,b) is any subinterval, and N any number, we can choose l s.t. $\frac{1}{p_N^l} < (b-a)$. And then we can choose a $k \in \mathbb{N}$ s.t. $\frac{k}{p_N^l} \in (a,b)$, and then $f_N\left(\frac{k}{p_N^l}\right) = 2 > 1$.

5.6

① We have $f((-1,1)) = [0,1)$ which is not open.

② Let $u \in B(0, qs)$, i.e. $\|u\| < qs \Leftrightarrow \frac{\|u\|}{q} < s \Leftrightarrow \|\frac{u}{q}\| < s$.

Let $v = \frac{u}{q}$. Then $v \in B(0,s)$, so $v \in \overline{A(B(0,r))}$ by assumption.

Can then find $v_n \in A(B(0,r))$ s.t $v_n \to v$, and therefore $t_n \in B(0,r)$ s.t. $A(t_n) = v_n$, $A(t_n) \to v$. Let $p_n = q t_n$. Then $\|p_n\| = \|q t_n\| = q\|t_n\| < qr$, so $p_n \in B(0,qr)$. Let $u_n = A(p_n)$. Then we have:

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} A(p_n) = \lim_{n\to\infty} A(q t_n) = \lim_{n\to\infty} q A(t_n) = q \lim_{n\to\infty} A(t_n)$$
$$= q \lim_{n\to\infty} v_n = q v = q \frac{u}{q} = u.$$

So $u \in \overline{A(B(0,qr))}$ as wanted.