5.6

(3)

$G(A) = \{(x, A(x)), x \in X\}$.

Let $u, v \in G(A)$. $u = (x, A(x))$, $v = (y, A(y))$.
Then $u+v = (x, A(x)) + (y, A(y)) = (x+y, A(x)+A(y)) = (x+y, A(x+y)) \in G(A)$,
and $\alpha u = \alpha(x, A(x)) = (\alpha x, \alpha A(x)) = (\alpha x, A(\alpha x)) \in G(A)$.
So $G(A)$ is a sub-space of $X \times Y$.

(4) The norm on $X \times Y$ is given by $\|(x,y)\|_{X\times Y} = \|x\|_X + \|y\|_Y$.

So $\|\pi(x, A(x))\|_X = \|x\|_X \leq \|x\|_X + \|A(x)\|_Y = \|(x, A(x))\|_{X\times Y}$.
π is therefore Lipschitz, and Lipschitz $\Rightarrow$ continuous.
$\pi((x+y, A(x+y))) = x+y = \pi((x, A(x))) + \pi((y, A(y)))$, so π is linear.
As $x \mapsto (x, A(x))$ is linear, continuous, it's bounded, i.e.
there exists a K s.t. $\|(x, A(x))\|_{X\times Y} \leq K\|x\|$, and then we have
$\|A(x)\|_Y \leq \|x\|_X + \|A(x)\|_Y = \|(x, A(x))\|_{X\times Y} \leq K\|x\|$, so A is bounded,
hence continuous.

(5) $|\cdot|$ and $\|\cdot\|$ are equivalent if there exists a C and a D s.t.
$|x| \leq C\|x\|$ and $\|x\| \leq D|x|$.
We already know that C exists, so must find D.
Let $id: (V, \|\cdot\|) \to (V, |\cdot|)$ be given by $id(x) = x$.
Then id is linear, surjective and bounded ($|x| = |id(x)| \leq C \cdot \|x\|$),
so by the open mapping theorem, id is open.
Then $(id^{-1})^{-1}(U) = id(U)$ is open whenever U is open, so
id^{-1} is continuous, and therefore bounded. So there must
exist a $D = \|id^{-1}\|$ s.t. $\|x\| = \|id^{-1}(x)\| \leq D|x|$, as wanted.

(6) Note: Should be unique $y \in Y$ s.t. $A(x) = B(y)$.

C is linear: Let $x_1, x_2 \in X$ with $y_1, y_2 \in Y$ s.t. $A(x_1) = B(y_1)$ and $A(x_2) = B(y_2)$.

Then $A(x_1 + x_2) = A(x_1) + A(x_2) = B(y_1) + B(y_2) = B(y_1 + y_2)$

so $C(x_1 + x_2) = y_1 + y_2 = C(x_1) + C(x_2)$

Also, $A(\alpha x_1) = \alpha A(x_1) = \alpha B(y_1) = B(\alpha y_1)$, so $C(\alpha x_1) = \alpha y_1 = \alpha C(x_1)$.

C is bounded $\Leftrightarrow$ $G(C)$ is closed:

As A is bounded, $G(A)$ is closed.

Let $D(x, y) = (x, B(y))$. Then D is linear and bounded, so D is continuous.

Therefore $D^{-1}(G(A))$ is closed.

But $D^{-1}(G(A)) = \{(x,y) \mid D(x,y) \in \{(x_1, A(x_1)) \mid x_1 \in X\}\}$

$= \{(x,y) \mid$ there exists an $x_1 \in X$ s.t. $x = x_1$ and $B(y) = A(x_1)\}$

$= \{(x,y) \mid$ there exists $x_1 \in X$, $x = x_1$, $y = C(x)\}$

$= \{(x, C(x)) \mid x \in X\} = G(C)$, so $G(C)$ is closed.

(6) Note: Should be unique $y \in Y$ s.t. $A(x) = B(y)$.

C is linear: Let $x_1, x_2 \in X$, $y_1, y_2 \in Y$ s.t. $A(x_1) = B(y_1)$, $A(x_2) = B(y_2)$.

Then: $A(x_1 + x_2) = A(x_1) + A(x_2) = B(y_1) + B(y_2) = B(y_1 + y_2)$

so $C(x_1 + x_2) = y_1 + y_2 = C(x_1) + C(x_2)$,

and $A(\alpha x_1) = \alpha A(x_1) = \alpha B(y_1) = B(\alpha y_1)$,

so $C(\alpha x_1) = \alpha y_1 = \alpha C(x_1)$, as wanted.

C is bounded $\Leftrightarrow$ $G(C)$ is closed:

Let (x_n, y_n) be a sequence in $G(C)$ converging to $(x, y) \in X \times Y$.

Then we have $x_n \to x$ and $y_n \to y$, and $A(x_n) = B(y_n)$.

As both A and B are continuous, we have

$A(x) = A(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} A(x_n) = \lim_{n\to\infty} B(y_n) = B(\lim_{n\to\infty} y_n) = B(y)$,

so $C(x) = y$. Therefore, $(x, y) \in G(C)$, so $G(C)$ is closed.

Note: Should be $a > 0$

(7) A bounded below $\Rightarrow$ $A(X)$ closed subspace of Y:

$A(X)$ is always a subspace of Y as if $y_1, y_2 \in A(X)$, there exists $x_1, x_2 \in X$ s.t. $A(x_1) = y_1$, $A(x_2) = y_2$. We have $y_1 + y_2 = A(x_1) + A(x_2) = A(x_1 + x_2) \in A(X)$, and $\alpha y_1 = \alpha A(x_1) = A(\alpha x_1) \in A(X)$, so $A(X)$ is a subspace.

Now let $\{y_n\}$ be a sequence in $A(X)$ converging to $y \in Y$.

Then there exists $x_n \in X$ s.t. $A(x_n) = y_n$. As y_n is Cauchy, we have

$\frac{\varepsilon}{a} > \|y_n - y_m\| = \|A(x_n) - A(x_m)\| = \|A(x_n - x_m)\| \geq a\|x_n - x_m\|$ whenever $n, m \geq N$.

So $\|x_n - x_m\| < \varepsilon$ whenever $n, m \geq N$, and is therefore Cauchy.

X is complete, so $x_n \to x$. Then $y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} A(x_n) = A(\lim_{n\to\infty} x_n) = A(x)$,

so $y \in A(X)$, and $A(X)$ must therefore be bounded.

7 cont.

<u>$A(X)$ closed subspace of $Y \Rightarrow A$ bounded below:</u>

Let $A': X \to A(X)$ be given by $A'(x) = A(x)$. As A is injective, A' is surjective. As $A(X)$ is closed in Y, $A(X)$ is complete, so by the bounded inverse theorem, $A'^{-1}: A(X) \to X$ is bounded, $\|A'^{-1}(y)\| \leq C\|y\|$ for all $y \in A(X)$. For any $x \in X$, let $y = A(x)$. Then $\|A'^{-1}(y)\| = \|A'^{-1}(A(x))\| = \|x\| \leq C\|y\| = C\|A(x)\|$, and then we have $\frac{1}{C}\|x\| \leq \|A(x)\|$, so A is bounded below.

6.11

① Must check that $\vec{\sigma}(r) = \vec{F}(a+r) - \vec{F}(a) - \vec{O}(r)$ satisfies $\lim_{r \to 0} \frac{\|\sigma(r)\|}{\|r\|} = 0$.

As $\vec{F}$ is constant, $\vec{F}(a+r) - \vec{F}(a) = 0$, and $\vec{O}(r) = 0$, so $\sigma(r) = 0$. Therefore, $\lim_{r \to 0} \frac{\|\sigma(r)\|}{\|r\|} = \lim_{r \to 0} \frac{0}{\|r\|} = 0$, as wanted.

② Must check that $\sigma(r) = F(a+r) - F(a) - A(r)$ satisfies

$$\lim_{r \to 0} \frac{\|\sigma(r)\|}{\|r\|} = 0.$$

We have $F(a+r) = A(a+r) + C = A(a) + A(r) + C$
and $F(a) = A(a) + C$, so
$\sigma(r) = A(a) + A(r) + C - A(a) - C - A(r) = 0$,
so $\lim_{r \to 0} \frac{\|\sigma(r)\|}{\|r\|} = 0$.

③ $\sigma(r) = H(a+r) - H(a) - H'(a)$

$$= \alpha F(a+r) + \beta G(a+r) - \alpha F(a) - \beta G(a) - \alpha F'(a)(r) - \beta G'(a)(r)$$

$$= \alpha\left(F(a+r) - F(a) - F'(a)(r)\right) + \beta\left(G(a+r) - G(a) - G'(a)(r)\right)$$

$$= \alpha\, \sigma_F(r) + \beta \sigma_G(r) \text{ with } \lim_{r \to 0} \frac{\|\sigma_F(r)\|}{\|r\|} = \lim_{r \to 0} \frac{\|\sigma_G(r)\|}{\|r\|} = 0.$$

So $\lim_{r \to 0} \frac{\|\sigma(r)\|}{\|r\|} = \lim_{r \to 0} \frac{\|\alpha \sigma_F(r) + \beta \sigma_G(r)\|}{\|r\|} \leq \lim_{r \to 0} \frac{|\alpha|\|\sigma_F(r)\| + |\beta|\|\sigma_G(r)\|}{\|r\|}$

$= \lim_{r \to 0} \alpha \frac{\|\sigma_F(r)\|}{\|r\|} + \lim_{r \to 0} \beta \frac{\|\sigma_G(r)\|}{\|r\|} = 0 + 0 = 0$, as wanted.

4. Will show $C(\alpha x+\beta y)=\alpha C(x)+\beta C(y)$.

$$C(\alpha x+\beta y)=A(B(\alpha x+\beta y))=A(\alpha B(x)+\beta B(y))$$
$$=\alpha A(B(x))+\beta A(B(y))=\alpha C(x)+\beta C(y).$$

5. For $K=F\circ G\circ H$, use chain rule on $F\circ(G\circ H)$:

$$K'(a)=(F\circ(G\circ H))'(a)=F'(G\circ H(a))\circ(G\circ H)'(a)$$
$$=F'(c)\circ(G\circ H)'(a)=F'(c)\circ G'(H(a))\circ H'(a)$$
$$=F'(c)\circ G'(b)\circ H'(a)$$

For general n: Let $F_n: X_n\to X_{n+1}$, $x_{n+1}=F_n(x_n)$, $x_1\in X_1$

Assume F_n differentiable at x_n.

Theorem: Let $F=F_n\circ F_{n-1}\circ\cdots\circ F_1: X_1\to X_{n+1}$.

Then $F'(x_1)=F_n'(x_n)\circ F_{n-1}'(x_{n-1})\circ\cdots\circ F_1'(x_1)$

Proof: By induction, true for $n=2,3$.

Assume true for $n=k-1$, i.e. $(F_{n-1}\circ\cdots\circ F_1)'(x_1)=F_{n-1}'(x_{n-1})\circ\cdots\circ F_1'(x_1)$.

Use chain rule on $F_n\circ(F_{n-1}\circ\cdots\circ F_1)$:

$$F'(x_1)=F_n'(F_{n-1}\circ\cdots\circ F_1(x_1))\circ(F_{n-1}\circ\cdots\circ F_1)'(x_1)$$
$$=F_n'(x_n)\circ F_{n-1}'(x_{n-1})\circ\cdots\circ F_1'(x_1)$$ as wanted.

⑦ Will use that $F(a)(r) = F(a;r) = \lim_{t\to 0} \frac{F(a+tr)-F(a)}{t}$

Choose $r = e_i$. Then $\frac{F(a+tr)-F(a)}{t} = \frac{(f_1(a+te_i),\dots,f_m(a+te_i)) - (f_1(a),\dots f_m(a))}{t}$

$$= \left(\frac{f_1(a+te_i)-f_1(a)}{t}, \dots, \frac{f_m(a+te_i)-f_m(a)}{t}\right)$$

As F is differentiable, the limit as $t\to 0$ exists, by thm 6.1.11, and we recognize the limit as $\left(\frac{\partial f_1}{\partial x_i}, \frac{\partial f_2}{\partial x_i}, \dots, \frac{\partial f_m}{\partial x_i}\right)$.

So $F(a;e_i) = \begin{pmatrix} \frac{\partial f_1}{\partial x_i} \\ \vdots \\ \frac{\partial f_m}{\partial x_i} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_i} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_i} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = J(a) e_i.$

Then, for any $r = \alpha_1 e_1 + \dots + \alpha_m e_m$

$$\begin{aligned} F(a;r) &= F(a)(r) = F(a)(\alpha_1 e_1 + \dots + \alpha_m e_m) = \alpha_1 F(a)(e_1) + \dots + \alpha_m F(a)(e_m) \\ &= \alpha_1 J(a) e_1 + \dots + \alpha_m J(a) e_m \\ &= J(a)(\alpha_1 e_1 + \dots + \alpha_m e_m) = J(a) r, \text{ as wanted.} \end{aligned}$$

By uniqueness of $F(a)$, we have $F(a) = J(a)$.

⑧

Must show $h(x+r)-h(x)-h'(x)r = \sigma(r)$ satisfies $\lim_{\|r\|_X \to 0} \frac{|\sigma(r)|}{\|r\|_X} = 0$.

$$h(x+r)-h(x)-h'(x)r = \langle F(x+r), G(x+r)\rangle - \langle F(x), G(x)\rangle - \langle F'(x)r, G(x)\rangle - \langle F(x), G'(x)r\rangle$$

$$= \langle F(x+r), G(x+r)\rangle - \langle F(x)+F'(x)r, G(x)\rangle - \langle F(x), G'(x)r\rangle$$

$$= \langle F(x+r), G(x+r)\rangle - \langle F(x)+F'(x)r, G(x)+G(x+r)-G(x+r)\rangle - \langle F(x), G'(x)r\rangle$$

$$= \langle F(x+r)-F(x)-F'(x)r, G(x+r)\rangle + \langle F(x)+F'(x)r, G(x+r)-G(x)\rangle - \langle F(x), G'(x)r\rangle$$

$$= \langle \sigma_F(r), G(x+r)\rangle + \langle F(x)+F'(x)r, G(x+r)-G(x)\rangle - \langle F(x)+F'(x)r-F'(x)r, G'(x)r\rangle$$

$$= \langle \sigma_F(r), G(x+r)\rangle + \langle F(x)+F'(x)r, G(x+r)-G(x)-G'(x)r\rangle + \langle F'(x)r, G'(x)r\rangle$$

$$= \langle \sigma_F(r), G(x+r)\rangle + \langle F(x)+F'(x)r, \sigma_G(r)\rangle + \langle F'(x)r, G'(x)r\rangle$$

We have $|\langle F'(x)r, G'(x)r\rangle| \le \|F'(x)r\|_Y \cdot \|G'(x)r\|_Y \le \|F'(x)\|\|r\|_X \cdot \|G'(x)\|\|r\|_X$,

so $\lim_{\|r\|_X \to 0} \frac{|\langle F'(x)r, G'(x)r\rangle|}{\|r\|_X} \le \lim_{\|r\|_X\to 0} \|F'(x)\|\|G'(x)\|\|r\| = 0$, so

$$\lim_{\|r\|_X\to 0} \frac{|\sigma(r)|}{\|r\|_X} = \lim_{\|r\|_X\to 0} \frac{|\langle \sigma_F(r), G(x+r)\rangle + \langle F(x)+F'(x)r, \sigma_G(r)\rangle + \langle F'(x)r, G'(x)r\rangle|}{\|r\|_X}$$

$$\le \lim_{\|r\|_X\to 0} \frac{\|\sigma_F(r)\|\|G(x+r)\|_Y}{\|r\|_X} + \lim_{\|r\|_X\to 0} \frac{\|F(x)+F'(x)r\|\|\sigma_G(r)\|_Y}{\|r\|_X} + \lim_{\|r\|_X\to 0} \frac{|\langle F'(x)r, G'(x)r\rangle|}{\|r\|_X}$$

$= 0\,\|G(x)\| + \|F(x)\| \cdot 0 + 0 = 0$, as wanted.

(10)

We have $F(x+r) - F(x) = F'(x)(r) + \sigma(r)$.

As $F(x)$ is a maximum, we have $F(x+r) - F(x) \leq 0$ and $F(x-r) - F(x) \leq 0$.

So $F'(x)(r) + \sigma(r) \leq 0$ and $F'(x)(-r) + \sigma(r) = -F'(x)(r) + \sigma(r) \leq 0$.

Therefore $\lim_{\|r\| \to 0} \frac{F'(x)(r)}{\|r\|} \leq \lim_{\|r\| \to 0} \frac{\sigma(r)}{\|r\|} = 0$ and $\lim_{\|r\| \to 0} \frac{F'(x)(r)}{\|r\|} \geq \lim_{\|r\| \to 0} \frac{\sigma(-r)}{\|r\|} = 0$,

i.e. $\lim_{\|r\| \to 0} \frac{F'(x)(r)}{\|r\|} = \lim_{t \to 0} \frac{F'(x)(tr)}{\|tr\|} = 0$ for all r.

$$\lim_{t \to 0} \frac{F'(x)(tr)}{\|tr\|} = \lim_{t \to 0} \frac{\cancel{t} F'(x)(r)}{\cancel{t}\|r\|} = \frac{F'(x)(r)}{\|r\|} = 0, \text{ so } F'(x)(r) = 0.$$

(11) Enough to show that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist, if $r = (a, b)$, then $F'(\bar{0}, r) = a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y}$.

$$\lim_{h \to 0} \frac{f(0+h, 0) - f(0,0)}{h} = \lim_{h \to 0} \frac{\frac{(0+h)^2 \cdot 0}{(0+h)^4 + 0^2} - 0}{h} = \lim_{h \to 0} \frac{0}{h^5} = 0.$$

$$\lim_{h \to 0} \frac{f(0, 0+h) - f(0,0)}{h} = \lim_{h \to 0} \frac{\frac{0^2 (0+h)}{0^4 + (0+h)^2} - 0}{h} = \lim_{h \to 0} \frac{0}{h^3} = 0.$$

So both $\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = 0$ and $\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = 0$ exist.

On the curve $r(t) = (t, t^2)$ we have $f(r(t)) = \begin{cases} \frac{t^4}{2t^4} = \frac{1}{2} & t \neq 0 \\ 0 & t = 0 \end{cases}$

So $r(t)$ is continuous, but $f(r(t))$ is not. Therefore $f(x,y)$ cannot be continuous and therefore not differentiable.