6.2

1)

a) As $F'(x) = 0$, we have $\|F'(x)\| \leq 0$ on O.

So by corr. 6.2.3 we have $\|F(b) - F(a)\| \leq 0\|b - a\| = 0$ for all $a, b \in O$. So $F(b) - F(a) = 0 \Rightarrow F(a) = F(b)$ for all $a, b \in O$, i.e. F is constant.

b) Define $F(x) = H(x) - G(x)$. Then we have

$F'(x) = H'(x) - G'(x) = 0$. By a), $F(x)$ is constant,

so $H(x) - G(x) = C \Rightarrow H(x) = G(x) + C$.

c) Note: This was originally wrong, but has been fixed.

What does it mean for F' to be constant?

For every $x \in X$, $F'(x)$ is a linear operator from X to Y.

F' constant on O should then be that $F'(x)$ is the same linear operator as $F'(y)$ for all $x, y \in O$. Call this linear operator G.

As we have $G'(x) = G$ for all linear operators, we have that $F'(x) = G'(x)$ for all x in O, so by b) we have $F = G + C$ on O.

② For each interval $[t_i, t_{i+1}]$ we have by Thm 6.2.1 (Mean Value Theorem) that $\|F(t_{i+1}) - F(t_i)\| \leq g(t_{i+1}) - g(t_i)$

Also, for the intervals $[a, t_1]$ and $[t_n, b]$ we have that

$$\|F(t_1) - F(a)\| \leq g(t_1) - g(a) \quad \text{and} \quad \|F(b) - F(t_n)\| \leq g(b) - g(t_n)$$

So
$$\begin{aligned}\|F(b) - F(a)\| &= \|F(b) - F(t_n) + F(t_n) - F(t_{n-1}) + \dots + F(t_2) - F(t_1) + F(t_1) - F(a)\| \\ &\leq \|F(b) - F(t_n)\| + \|F(t_n) - F(t_{n-1})\| + \dots + \|F(t_2) - F(t_1)\| + \|F(t_1) - F(a)\| \\ &\leq g(b) - \cancel{g(t_n)} + \cancel{g(t_n)} - \cancel{g(t_{n-1})} + \dots + \cancel{g(t_2)} - \cancel{g(t_1)} + \cancel{g(t_1)} - g(a) \\ &= g(b) - g(a) \qquad \text{as wanted.}\end{aligned}$$

③ a) Define $G_{n,m}(x) = F_n(x) - F_m(x)$.

Then $\|F_n(x) - F_n(a) - (F_m(x) - F_m(a))\| = \|G_{n,m}(x) - G_{n,m}(a)\|$ and $\|G'(x)\| = \|F_n'(x) - F_m'(x)\| \leq \|F_n' - F_m'\|_\infty$, so by Corr 6.2.3 we have
$$\begin{aligned}\|G_{n,m}(x) - G_{n,m}(a)\| &\leq \|F_n' - F_m'\|_\infty \cdot \|x - a\| \\ &\leq \|F_n' - F_m'\|_\infty \cdot K\end{aligned}$$
as O is bounded.

b) Will first show that $F_n(x)$ is Cauchy for each $x \in O$.

We have
$$\begin{aligned}\|F_n(x) - F_m(x)\| &= \|F_n(x) - F_m(x) - (F_n(a) - F_m(a)) + (F_n(a) - F_m(a))\| \\ &\leq \|F_n(x) - F_n(a) - (F_m(x) - F_m(a))\| + \|F_n(a) - F_m(a)\| \\ &\leq K\|F_n' - F_m'\|_\infty + \|F_n(a) - F_m(a)\| \\ &< K\frac{\varepsilon}{2K} + \frac{\varepsilon}{2} = \varepsilon \qquad \text{when } n, m \geq N\end{aligned}$$
as F_n' converges uniformly and F_n converges in a.

As Y is complete, $F_n(x)$ converges for each x. Let $F(x) = \lim_{n \to \infty} F_n(x)$.

As the choice of N earlier was independent of x, we have

$\lim_{m \to \infty} \|F_n(x) - F_m(x)\| = \|F_n(x) - F(x)\| < \varepsilon$ for all x, when $n \geq N$, so F_n converges uniformly to F.

③

c) By Theorem 5.4.8 G is linear, by uniform convergence G is continuous and by Theorem 5.4.8 G is then bounded.

So by definition of the derivative, we have $F'(x) = G(x)$ if $\|\sigma(r)\| = \|F(x+r) - F(x) - G(x)(r)\|$ goes to zero faster than r for all $x \in O$.

d) $$\begin{aligned}\|F(x+r) - F(x) - G(x)(r)\| &= \|F(x+r) - F(x) - (F_n(x+r) - F_n(x)) + (F_n(x+r) - F_n(x)) - G(x)(r)\| \\ &\leq \|F(x+r) - F(x) - (F_n(x+r) - F_n(x))\| \\ &\quad + \|F_n(x+r) - F_n(x) - F_n'(x)(r) + F_n'(x)(r) - G(x)(r)\| \\ &\leq \|F(x+r) - F(x) - (F_n(x+r) - F_n(x))\| \\ &\quad + \|F_n(x+r) - F_n(x) - F_n'(x)(r)\| + \|F_n'(x)(r) - G(x)(r)\|\end{aligned}$$

e) We have $$\begin{aligned}\|F(x+r) - F(x) - (F_n(x+r) - F_n(x))\| &= \lim_{m\to\infty} \|F_m(x+r) - F_m(x) - (F_n(x+r) - F_n(x))\| \\ &\leq \|F_m' - F_n'\|_\infty \|x+r-x\| \quad \text{by Corr 6.2.3} \\ &\leq \frac{\varepsilon}{3}\|r\| \quad \text{when } n \geq N_1 \text{ as } F_n' \text{ is Cauchy.}\end{aligned}$$

f) As $F_n' \to G$, we have that $\|F_n - G\| < \frac{\varepsilon}{3}$ when $n \geq N$, and $F_n - G$ is a bounded linear function, so
$$\|(F_n - G)(r)\| \leq \|F_n - G\|\|r\| < \frac{\varepsilon}{3}\|r\|$$

g) By definition of F_n', $\|F_n(x+r) - F_n(x) - F_n'(x)(r)\|$ goes to zero faster than r, so if r is small enough we have
$$\frac{\|F_n(x+r) - F_n(x) - F_n'(x)(r)\|}{\|r\|} < \frac{\varepsilon}{3} \Rightarrow \|F_n(x+r) - F_n(x) - F_n'(x)(r)\| < \frac{\varepsilon}{3}\|r\|$$

h) We now have $\|F(x+r) - F(x) - G(x)(r)\| \leq \left(\frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}\right)\|r\| = \varepsilon\|r\|$,

so $$\frac{\|F(x+r) - F(x) - G(x)(r)\|}{\|r\|} < \varepsilon$$ when $\|r\| < \delta$, as wanted.

6.31

(1) We have $\frac{\partial f}{\partial x}(a)(r_i) = f_a^{x\,'}(a_x)(r_i) = \lim_{t\to 0} \frac{f_a^x(a_x + t r_i) - f_a^x(a_x)}{t}$

$$= \lim_{t\to 0} \frac{f(a_x + tr, a_y) - f(a_x, a_y)}{t}$$

For $r=1$ we then get $\frac{\partial f}{\partial x}(a) = \lim_{t\to 0} \frac{f(a_x+t, a_y) - f(a_x, a_y)}{t}$

which is just the standard $\frac{\partial f}{\partial x}$. Same for $\frac{\partial f}{\partial y}$.

(2)

a) Let $H^x_{(x_0,\lambda_0)}(x)$ be the function we get by fixing $\lambda = \lambda_0$

Then $\frac{\partial H}{\partial x}(x_0,\lambda_0) = H^{x\,'}_{(x_0,\lambda_0)}(x_0) = (F(x) + \lambda_0 G(x))'(x_0)$

$$= F'(x_0) + \lambda G'(x_0) \quad \text{By prop 6.1.7 and 6.1.8.}$$

Let $H^\lambda_{(x_0,\lambda_0)}(\lambda)$ be the function we get by fixing $x = x_0$.

Then $\frac{\partial H}{\partial \lambda}(x_0,\lambda_0) = H^\lambda_{(x_0,\lambda_0)}(\lambda_0) = (F(x_0) + \lambda G(x_0))'(\lambda_0)$

$$= 0 + G(x_0) = G(x_0)$$

as $F(x_0)$ is constant and $\lambda G(x_0)$ is the function (linear) given by multiplying by $G(x_0)$.

b) By thm 6.3.3 we have $H'(x,\lambda)(v) = \frac{\partial H}{\partial x}(x,\lambda)(r) + \frac{\partial H}{\partial \lambda}(x,\lambda)(v)$, and by exercise 6.1.10 we have $H'(x,\lambda)(v) = 0$.

So we have $F'(x) + \lambda G'(x) + G(x) = 0$,

and $G(x) = 0$ by assumption,

Therefore we have $F'(x) + \lambda G'(x) = 0$

③ When y is held still, $x \mapsto \langle x, y\rangle$ is a linear operator, and for all linear operators we have $G'(x)(r) = G(r)$, so we have

$\frac{\partial F}{\partial x}(r) = \langle r, y\rangle$. As it's a real inner space, $\langle x, y\rangle = \langle y, x\rangle$, so by the same reasoning $\frac{\partial F}{\partial y}(x,y)(s) = \langle s, x\rangle = \langle x, s\rangle$.

④ From Prop. 6.3.2 we have

$$\frac{\partial G}{\partial x}(a,b)(r) = G'(a,b)(r,0)$$
$$= J_G \cdot \begin{pmatrix} r \\ 0 \end{pmatrix}$$
$$= \left(\frac{\partial G}{\partial x_1}(a,b), \dots, \frac{\partial G}{\partial x_n}(a,b), \frac{\partial G}{\partial y_1}(a,b), \dots \frac{\partial G}{\partial y_m}(a,b)\right) \cdot \begin{pmatrix} r \\ 0 \end{pmatrix}$$
$$= \left(\frac{\partial G}{\partial x_1}(a,b), \dots, \frac{\partial G}{\partial x_n}(a,b)\right) \cdot r$$

By the same reasoning, $\frac{\partial G}{\partial y}(a,b)(r) = \left(\frac{\partial G}{\partial y_1}(a,b), \dots, \frac{\partial G}{\partial y_m}(a,b)\right) \cdot r$

⑤ If we keep f constant, we know from the fundamental theorem of calculus that $F_f(t) = \int_0^t f(s)ds$ satisfies $F_f'(t) = f(t)$.
If we keep t constant, $F_t(f) = \int_0^t f(s)ds$ is a linear function, so is it's own derivative, i.e. $F_t'(f)(g) = F_t(g) = i(g)(t)$.

⑥ If we keep f constant, $F_f'(t) = (f(t))' = f'(t)$.
If we keep t constant, we have that $F_t(f)$ is a linear function, so it's its own derivative, i.e. $F_t'(f)(g) = F_t(g) = \varepsilon_t(g)$.

6.41

① There (was?) is an error in the definition of $\int_a^b F(t)dt$ when $a > b$. We want $\int_a^b F(t)dt = -\int_b^a F(t)dt$

<u>Then:</u>

- Order $a < c < b$: From proposition.
- Order $a < b < c$: $$\int_a^b F(x)dx = \underbrace{\int_a^b F(x)dx + \int_b^c F(x)dx}_{\int_a^c} - \int_b^c F(x)dx$$ $$= \int_a^c F(x)dx + \int_c^b F(x)dx \quad \boxed{\checkmark}$$
- Order $b < c < a$: $$\int_a^b F(x)dx = -\int_b^a F(x)dx = -\int_b^c F(x)dx - \int_c^a F(x)dx$$ $$= \int_a^c F(x)dx + \int_c^b F(x)dx$$
- Order $b < a < c$: $$\int_a^b F(x)dx = -\int_b^a F(x)dx = -\int_b^a F(x)dx - \int_a^c F(x)dx + \int_a^c F(x)dx$$ $$= -\left(\int_b^a F(x)dx + \int_a^c F(x)dx\right) + \int_a^c F(x)dx)$$ $$= -\int_b^c F(x)dx + \int_a^c F(x)dx$$ $$= \int_a^c F(x)dx + \int_c^b F(x)dx$$
- Order $c < a < b$: $$\int_a^b F(x)dx = \int_a^b F(x)dx + \int_c^a f(x)dx - \int_c^a f(x)dx$$ $$= \int_c^b F(x)dx - \int_c^a F(x)dx = \int_a^c F(x)dx + \int_c^b F(x)dx$$
- Order $c < b < a$: $$\int_a^b F(x)dx = -\int_b^a F(x)dx + \int_c^b F(x)dx - \int_c^b F(x)dx$$ $$= -\left(\int_c^b F(x)dx + \int_b^a F(x)dx\right) + \int_c^b F(x)dx$$ $$= -\int_c^a F(x)dx + \int_c^b F(x)dx = \int_a^c F(x)dx + \int_c^b F(x)dx.$$

② For $r<0$ we still have

$$I(x+r)-I(x)=\int_a^{x+r}F(t)\,dt-\int_a^{x}F(t)\,dt$$

$$=\cancel{\int_a^{x}F(t)\,dt}+\int_x^{x+r}F(t)\,dt-\cancel{\int_a^{x}F(t)\,dt}$$

$$=\int_x^{x+r}F(t)\,dt \quad \text{by exercise ①.}$$

So $\sigma(r)=\int_x^{x+r}F(t)\,dt-F(x)\cdot r=\int_x^{x+r}F(t)-F(x)\,dt$

$$=-\int_{x+r}^{x}F(t)-F(x)\,dt$$

So we can get $\|F(t)-F(x)\|<\varepsilon$ when $|t-x|<r<\delta$, so we have $\|\sigma(r)\|<\varepsilon r$ as wanted.

③ For every Riemann-sum we have

$$\|R(F,\Pi,S)\|=\left\|\sum_{i=1}^{n}F(c_i)(x_{i+1}-x_i)\right\|\le\sum_{i=1}^{n}\|F(c_i)\|(x_{i+1}-x_i)=R(\|F\|,\Pi,S),$$

so $\left\|\int_a^b F(x)\,dx\right\|\le\int_a^b\|F(x)\|\,dx.$

So! $\|F(b)-F(a)\|=\left\|\int_a^b F'(x)\,dx\right\|\le\int_a^b\|F'(x)\|\,dx\le\int_a^b g'(x)\,dx=g(b)-g(a)$

④ Let $G(t)=\int_a^t F(x)\,dx$. Then $G'(t)=F(t)$ by Thm 6.4.6,

$G(a)=\int_a^a F(x)\,dx=0$ and as G is differentiable, G is continuous.

If there is another $H(t)$ s.t. $H'(t)=F(t)$ we know that we must have $G(t)=H(t)+C$. But if $H(a)=0$, then we have $G(a)=H(a)+C \Rightarrow C=0$, so $G(t)=H(t)$. So G is unique.

(5)

If $a > b$ we have $\int_a^b g'(t)F'(g(t))\,dt = -\int_b^a g'(t)F'(g(t))\,dt$

$$= -\left(F(g(a)) - F(g(b))\right)$$

$$= F(g(b)) - F(g(a))$$

so if it's true for $a < b$ it will be true for $a > b$.

Assume $a < b$.

We have from Corr 6.4.7 and Thm 6.1.9 that

$$F \circ g(b) - F \circ g(a) = \int_a^b (F \circ g)'(t)\,dt$$

$$= \int_a^b F'(g(t)) \cdot g'(t)\,dt \quad \text{as wanted.}$$