6.5

(1)

Have: $F(0,0) = (1,-2)$,
$$JF = \begin{pmatrix} 2x, & 1 \\ 1, & -1 \end{pmatrix},$$

$JF(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$, which is invertible.

So $F'(0,0)$ is continuous and invertible, by the inverse function theorem we then have a G defined around $F(0,0) = (1,-2)$ with $G = F^{-1}$.

Also have: $F(-1,-1) = ((-1)^2 - 1 + 1, -1 + 1 - 2) = (1, -2)$

$JF(-1,-1) = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}$, which is invertible.

So $F'(-1,-1)$ is continuous and invertible, by the inverse function theorem, we then have a H defined around $F(-1,-1) = (1,-2)$ with $H = F^{-1}$.

We have $G'(1,-2) = (F'(0,0))^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix}$

and $H'(1,-2) = (F'(-1,-1))^{-1} = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix}$.

2a)

$$\left(\begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{cccccc} 2 & 0 & 2 & 2 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{cccccc} 3 & 0 & 0 & 2 & 0 & 1 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{cccccc} 3 & 0 & 0 & 2 & 0 & 1 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 3 & 0 & -6 & 0 & 0 & 3 \end{array}\right) \sim$$

$$\left(\begin{array}{cccccc} 3 & 0 & 0 & 2 & 0 & 1 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -6 & -2 & 0 & 2 \end{array}\right) \sim$$

$$\left(\begin{array}{cccccc} 3 & 0 & 0 & 2 & 0 & 1 \\ 12 & 6 & 6 & 0 & 6 & 0 \\ 0 & 0 & 6 & 2 & 0 & -2 \end{array}\right) \sim$$

$$\left(\begin{array}{cccccc} 3 & 0 & 0 & 2 & 0 & 1 \\ 12 & 6 & 0 & -2 & 6 & 2 \\ 0 & 0 & 6 & 2 & 0 & -2 \end{array}\right) \sim$$

$$\left(\begin{array}{cccccc} 3 & 0 & 0 & 2 & 0 & 1 \\ 0 & 6 & 0 & -10 & 6 & -2 \\ 0 & 0 & 6 & 2 & 0 & -2 \end{array}\right) \sim$$

$$\left(\begin{array}{cccccc} 1 & 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{5}{3} & 1 & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} & 0 & -\frac{1}{3} \end{array}\right)$$

So $A^{-1} = \begin{pmatrix} \frac{2}{3} & 0 & \frac{1}{3} \\ -\frac{5}{3} & 1 & -\frac{1}{3} \\ \frac{1}{3} & 0 & -\frac{1}{3} \end{pmatrix}$

2b)

$$JF(x,y,z) = \begin{pmatrix} 1 & 0 & 1 \\ 2x & y & 1 \\ 1 & 0 & 2z \end{pmatrix}$$

c) $F(1,1,-1) = \begin{pmatrix} 0 \\ 1^2 + \frac{1}{2} - 1 \\ 1 + (-1)^2 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 2 \end{pmatrix}$

and $JF(1,1,-1) = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix} = A$

So $F'(1,1,-1)$ is invertible and continuous, and by the inverse function theorem there exists a G defined around $F(1,1,-1) = (0,\frac{1}{2},2)$ with $G = F^{-1}$ and

$$G'(0,\tfrac{1}{2},2) = (F'(1,1,-1))^{-1} = A^{-1} = \begin{pmatrix} 2/3 & 0 & 1/3 \\ -5/3 & 1 & -1/3 \\ 1/3 & 0 & -1/3 \end{pmatrix}$$

3) Assume that F has a differentiable local inverse G with $n \neq m$.

Then $F \circ G = Id$, which gives us

$$(F \circ G)'(x) = Id'(x) = Id$$
$$\shortparallel$$
$$F'(G(x)) \cdot G'(x)$$

which gives $JF(G(x)) \circ JG(x) = I_m$

Also $G \circ F = Id$, so $JG(F(y)) \circ JF(y) = I_n$

But then for $y = G(x)$ $JF(y)$ is an invertible matrix for $n \neq m$, which is impossible!

(4)

Choose any point $y \in F(O)$. Then there exists an $x \in O$ s.t. $F(x) = y$, and as $F'(x)$ is invertible there exist an open $U_0 \subseteq O$ with $x \in U_0$ and $V_0 \subseteq Y$ with $y \in V_0$ s.t. F is a bijection from U_0 to V_0.

Therefore $V_0 \subseteq F(O)$, so for any $y \in F(O)$ there exist an open set V_0 with $y \in V_0 \subseteq F(O)$, and therefore a ball $B(y, \varepsilon) \subseteq V_0 \subseteq F(O)$ exists. So y is an interior point, and $F(O)$ is open.

⑤ We try using the directional derivatives to find ~~the true~~ derivative.

a)
$$P_n'(A;R) = \lim_{t\to 0} \frac{P_n(A+tR) - P_n(A)}{t} = \lim_{t\to 0} \frac{(A+tR)^n - A^n}{t}$$
$$= \lim_{t\to 0} \frac{A^n + \sum_{i=0}^{n-1} tA^i R A^{n-1-i} + t^2 O(A,R) - A^n}{t}$$
$$= \lim_{t\to 0} \sum_{i=0}^{n-1} A^i R A^{n-i} + t\, O(A,R) = \sum_{i=0}^{n-1} A^i R A^{n-i}$$

where $t^2 O(A,R)$ is all the parts of $(A+tR)^n$ where you have at least two Rs in the products.

So $P_n'(A;R) = RA^{n-1} + ARA^{n-2} + A^2RA^{n-3} + \dots + A^{n-2}RA + A^{n-1}R$

We therefore assume that $P_n'(A)(R) = \sum_{i=0}^{n-1} A^i R A^{n-i}$ as well, and check if ~~this~~ is is true:

$$\|\sigma(R)\| = \| P_n(A+R) - P_n(A) - \sum_{i=0}^{n-1} A^i R A^{n-i} \|$$
$$= \| (A+R)^n - A^n - \sum_{i=0}^{n-1} A^i R A^{n-i} \| = \| A^n + \sum_{i=0}^{n-1} A^i R A^{n-i} + N(A,R) - A^n - \sum_{i=0}^{n-1} A^i R A^{n-i} \|$$
$$= \|N(A,R)\|$$
where $N(A,R)$ is a sum of products of A and R with at least two Rs in each product.

As $\|AB\| \le \|A\|\cdot\|B\|$ and $\|A+B\| \le \|A\| + \|B\|$ we have that $\|N(A,R)\| \le \|R\|^2 \cdot f(\|A\|, \|R\|)$ where $f(\|A\|,\|R\|) = \sum_{i=2}^{n} \|A\|^{n-i} \|R\|^{i-2}$,

so $\frac{\|\sigma(R)\|}{\|R\|} \le \frac{\|R\|^2 f(\|A\|,\|R\|)}{\|R\|} = \|R\| f(\|A\|,\|R\|)$, and $\lim_{\|R\|\to 0} \frac{\|\sigma(R)\|}{\|R\|} = 0$, as wanted.

(v) We have $M_n = \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, so it is complete.

Want to show that $S_k(A) = \sum_{n=0}^{k} \frac{A^n}{n!}$ is Cauchy.

<u>Have:</u> $\|S_k - S_l\| = \left\| \sum_{n=0}^{k} \frac{A^n}{n!} - \sum_{n=0}^{l} \frac{A^n}{n!} \right\| = \left\| \sum_{n=l+1}^{k} \frac{A^n}{n!} \right\| \leq \sum_{n=l+1}^{k} \frac{\|A\|^n}{n!}$

We have from calculus that for every $x \in \mathbb{R}$, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges.

Specifically $e^{\|A\|} = \sum_{n=0}^{\infty} \frac{\|A\|^n}{n!}$ is Cauchy, and we can find N

s.t. $\sum_{n=l+1}^{k} \frac{\|A\|^n}{n!} < \varepsilon$ whenever $l, k \geq N$, as wanted

c) We have $S_k'(A)(R) = \left(\sum_{n=0}^{k} \frac{P_n}{n!}\right)'(A)(R) = \sum_{n=0}^{k} \frac{P_n'(A)(R)}{n!}$

$$= \sum_{n=0}^{k} \frac{\sum_{i=0}^{n-1} A^i R A^{n-1-i}}{n!}$$

So $\lim_{k\to\infty} S_k'(A)(R) = \sum_{n=0}^{\infty} \frac{\sum_{i=0}^{n-1} A^i R A^{n-1-i}}{n!}$. If this converges uniformly,

we can use Exercise 6.2.3. to give us the derivative of exp.

Unfortunately, this convergence does not seem to be uniform.

<u>However!</u> Let $O_r = \{A \in M_n \mid \|A\| < r\}$

Then $\left\| \sum_{n=0}^{\infty} \frac{\sum_{i=0}^{n-1} A^i R A^{n-1-i}}{n!} - \sum_{n=0}^{k} \frac{\sum_{i=0}^{n-1} A^i R A^{n-1-i}}{n!} \right\| = \left\| \sum_{n=k+1}^{\infty} \frac{\sum_{i=0}^{n-1} A^i R A^{n-1-i}}{n!} \right\|$

$$\leq \sum_{n=k+1}^{\infty} \frac{\sum_{i=0}^{n-1} \|A\|^i \|R\| \|A^{n-1-i}\|}{n!} = \sum_{n=k+1}^{\infty} \frac{n\|A\|^{n-1}\|R\|}{n!} = \|R\| \sum_{n=k+1}^{\infty} \frac{\|A\|^{n-1}}{(n-1)!}$$

$$\leq \|R\| \sum_{n=k+1}^{\infty} \frac{r^{n-1}}{(n-1)!} \leq \|R\| \varepsilon \text{ when } k \text{ large enough.}$$

So operator norm of $\lim_{k\to\infty} S_k'(A)$ is less than ε for all $A \in O_r$, and we <u>do</u> have uniform convergence on O_r. By Exercise 6.2.3 we then have $\exp'(A)(R) = \sum \frac{\sum_{i=0}^{n-1} A^i R A^{n-1-i}}{n!}$ on O_r for all r, so

$$\exp'(A)(R) = \sum_{n=0}^{\infty} \frac{\sum_{i=0}^{n-1} A^i R A^{n-1-i}}{n!} \text{ for all } A \in M_n.$$

Not the prettiest of derivatives.

<u>Note:</u> $\exp'(I_m)(R) = \sum_{n=0}^{\infty} \frac{\sum_{i=0}^{n-1} I_m^i R I_m^{n-1-i}}{n!} = \sum_{n=0}^{\infty} \frac{nR}{n!} = R \cdot \sum_{n=1}^{\infty} \frac{1}{(n-1)!} = eI_m R$

So $\exp'(I_m) = eI_m$.

d) We have $\exp(I_n) = \sum_{m=0}^{\infty} \frac{I_n^m}{m!} = \sum_{m=0}^{\infty} \frac{I_n}{m!} = I_n \sum_{m=0}^{\infty} \frac{1}{m!} = I_n e = eI_n$.

Then $\exp'(I_n) = \exp I_n = eI_n$, which is invertible, so

we have that a local inverse (log) is defined in a neighborhood.

We have $\log'(eI_n) = (\exp'(I_n))^{-1} = (eI_n)^{-1} = \frac{1}{e} I_n$.

7

a) For $x \neq 0$ we have $f'(x) = (x + x^2 \cos\frac{1}{x})' = 1 + 2x\cos\frac{1}{x} + \cancel{x^2} \sin\frac{1}{x} \cdot \frac{1}{\cancel{x^2}}$

$$= 1 + 2x\cos\frac{1}{x} + \sin\frac{1}{x}.$$

For $x=0$ we have $f'(0) = \lim_{t \to 0} \frac{f(t) - f(0)}{t} = \lim_{t \to 0} \frac{t + t^2 \cos\frac{1}{t}}{t}$

$$= \lim_{t \to 0} 1 + t\cos\frac{1}{t}$$

$$= 1 + \lim_{t \to 0} t\cos\frac{1}{t} = 1 \quad \text{as } |\cos\tfrac{1}{t}| < 1.$$

We have that $\lim_{x \to 0} f'(x)$ does not exist, as $\sin\frac{1}{x}$ will oscillate rapidly between -1 and 1 as $x \to 0$, so the derivative is not continuous.

b) First, a small theorem:

Any continuous, differentiable, injective function $f: (a,b) \to \mathbb{R}$ will have at most one peak, i.e. where $f'(x) = 0$, $f''(x) < 0$.

Proof: Assume x and y are two different peaks for f.

As f is injective, $f(x) \neq f(y)$. Assume for simplicity $f(x) < f(y)$ and $x < y$.

As x is a peak, if we choose a small ε, $f(x+\varepsilon) < f(x)$. So we have that $x + \varepsilon < y$ and $f(x+\varepsilon) < f(x) < f(y)$. By the intermediate value theorem we have that there must exist a $z \in [x+\varepsilon, y]$ with $f(z) = f(x)$, but this cannot be the case as f was injective. $\square$

Now we just need to note that $f'(\frac{2}{3\pi k}) = 0$ and $f''(\frac{2}{3\pi k}) < 0$ for all $k \in \mathbb{N}$.

So no matter how small ε is, f will have an infinite number of peaks in $[0, \varepsilon)$, and hence cannot be injective. Does not contradict inverse function theorem, as we need continuity of f', which we don't have.

6.6|

(2) Have $f(-1,2,0) = -1 \cdot 2^2 e^0 + 0 = -4$.

So we define $F((x,y),z) = xy^2e^z + z + 4$

Then $F((-1,2),0) = 0$ and $\frac{\partial F}{\partial z} = xy^2e^z + 1$,

so $\frac{\partial F}{\partial z}((-1,2),0) = -4+1 = -3$. We can find an open set around $(-1,2,0)$ where $\frac{\partial F}{\partial z} \neq 0$, so as a linear function it's bijective from $\mathbb{R}$ to $\mathbb{R}$.

Then by Implicit Function Theorem there exists a $g(x,y)$ s.t. $g(-1,2) = 0$ and $F((x,y), g(x,y)) = 0$, i.e. $f((x,y), g(x,y)) = -4$.

Have
$$g'(x,y) = \left(\frac{\partial g}{\partial x}(x,y), \frac{\partial g}{\partial y}(x,y)\right)$$
$$= -\left(\frac{\partial F}{\partial z}(x,y,g(x,y))\right)^{-1} \cdot \frac{\partial F}{\partial (x,y)}((x,y), g(x,y))$$
$$= -\left(xy^2 e^{g(x,y)} + 1\right)^{-1} \cdot \left(y^2 e^{g(x,y)}, 2xy e^{g(x,y)}\right)$$

$$g'(-1,2) = \left(\frac{\partial g}{\partial x}(-1,2), \frac{\partial g}{\partial y}(-1,2)\right)$$
$$= -(-3)^{-1} \cdot \left(4e^0, -4e^0\right) = \frac{1}{3}(4,-4) = \left(\frac{4}{3}, \frac{-4}{3}\right).$$

so $\frac{\partial g}{\partial x}(-1,2) = \frac{4}{3}$ and $\frac{\partial g}{\partial y} = \frac{-4}{3}$.

(3)

Let $F(x, y) = x^3 + y^3 + y - 1$.

Then $\frac{\partial F}{\partial y} = 3y^2 + 1 \neq 0$ for any (x_0, y_0) on the curve,

so we can by Implicit function theorem find an $f(x) = y$

s.t. $F(x, f(x)) = 0$ i.e. $x^3 + f(x)^3 + f(x) = 1$, i.e. $(x, f(x))$ satisfies the equation,

We have: $f'(x_0, y_0) = -\left(\frac{\partial F}{\partial y}(x_0, y_0)\right)^{-1} \cdot \left(\frac{\partial F}{\partial x}(x_0, y_0)\right)$

$$= -\left(3y_0^2 + 1\right)^{-1} \cdot \left(3x_0\right) = \frac{-3x_0}{3y_0^2 + 1}.$$

(4)

Define $F(x, y) = \phi(x, y) - C$

We assume $\frac{\partial F}{\partial y} = \frac{\partial \phi}{\partial y}$ is non-zero,

Then by implicit function theorem $y(x)$ exists, and

we have $y'(x) = -\left(\frac{\partial F}{\partial y}(x, y(x))\right)^{-1} \cdot \frac{\partial F}{\partial x}(x, y(x)) = -\frac{1}{\frac{\partial \phi}{\partial y}(x, y(x))} \cdot \frac{\partial \phi}{\partial x}(x, y(x))$

$$= -\frac{\frac{\partial \phi}{\partial x}(x, y(x))}{\frac{\partial \phi}{\partial y}(x, y(x))},$$ as wanted.

⑥

a) Define $F(x,t) = \phi(x, y(t))$

Then $\frac{\partial F}{\partial x}(x,t) = \frac{\partial \phi}{\partial x}(x, y(t))$. Assume $\frac{\partial \phi}{\partial x}(x, y(t)) \neq 0$ for x, t

By Implicit Function Theorem, there exist a $x(t)$ s.t.

$\phi(x(t), y(t)) = 0$, and we have that

$$x'(t) = -\left(\frac{\partial F}{\partial x}(x(t),t)\right)^{-1} \frac{\partial F}{\partial t}(x(t),t)$$

$$= -\left(\frac{\partial \phi}{\partial x}(x(t), y(t))\right)^{-1} \cdot \frac{\partial \phi}{\partial y}(x(t), y(t)) \cdot \frac{\partial y}{\partial t}(t)$$

$$= -\frac{\frac{\partial \phi}{\partial y}(x(t), y(t))}{\frac{\partial \phi}{\partial x}(x(t), y(t))} y'(t)$$ as wanted.

Assumptions: $\frac{\partial F}{\partial x}(x,t) = \frac{\partial \phi}{\partial x}(x,y) \neq 0$, and both

$\frac{\partial F}{\partial x}(x,t) = \frac{\partial \phi}{\partial x}(x,y)$ and $\frac{\partial F}{\partial t}(x,t) = \frac{\partial \phi}{\partial y}(x, y(t))\, y'(t)$

are existing and continuous.

b) We define $F(x,t) = \phi(x, y(t), z(t))$.

Assume $\frac{\partial F}{\partial x}(x,t) = \frac{\partial \phi}{\partial x}(x, y(t), z(t)) \neq 0$ for x, t.

By Implicit Function Theorem, there exist a $x(t)$ s.t.

$\phi(x(t), y(t), z(t)) = 0$ and we have that

$$x'(t) = -\left(\frac{\partial F}{\partial x}(x(t),t)\right)^{-1} \frac{\partial F}{\partial t}(x(t),t)$$

$$= -\left(\frac{\partial \phi}{\partial x}(x(t), y(t), z(t))\right)^{-1} \cdot \left(\frac{\partial \phi}{\partial y}(x(t), y(t), z(t))\, y'(t) + \frac{\partial \phi}{\partial z}(x(t), y(t), z(t))\, z'(t)\right)$$

$$= -\frac{\frac{\partial \phi}{\partial y} y'(t) + \frac{\partial \phi}{\partial z} z'(t)}{\frac{\partial \phi}{\partial x}} (x(t), y(t), z(t)).$$

⑦ We define $F(X,y,z)=\phi(X,y,z)$, $G(x,Y,z)=\phi(x,Y,z)$
and $H(x,y,Z)=\phi(x,y,Z)$.

Assume: $\dfrac{\partial F}{\partial X}=\dfrac{\partial\phi}{\partial x}(X,y,z)\neq 0$, $\dfrac{\partial G}{\partial y}=\dfrac{\partial\phi}{\partial y}(x,Y,z)\neq 0$, $\dfrac{\partial H}{\partial Z}=\dfrac{\partial\phi}{\partial z}(x,y,Z)\neq 0$.

By Implicit Function Theorem, we then have

$X(y,z)$, $Y(x,z)$, $Z(x,y)$ s.t. $\phi(X(y,z),y,z)=0$

$\phi(x,Y(x,z),z)=0$

$\phi(x,y,Z(x,y))=0$

with:
$$\frac{\partial X}{\partial(y,z)}=\left(\frac{\partial X}{\partial y},\frac{\partial X}{\partial z}\right)=-\left(\frac{\partial F}{\partial X}\right)^{-1}\frac{\partial F}{\partial(y,z)}$$
$$=-\left(\frac{\frac{\partial\phi}{\partial y}}{\frac{\partial\phi}{\partial x}},\frac{\frac{\partial\phi}{\partial z}}{\frac{\partial\phi}{\partial x}}\right)$$

$$\frac{\partial Y}{\partial(x,z)}=\left(\frac{\partial Y}{\partial x},\frac{\partial Y}{\partial z}\right)=-\left(\frac{\partial F}{\partial Y}\right)\frac{\partial F}{\partial(x,z)}$$
$$=-\left(\frac{\frac{\partial\phi}{\partial x}}{\frac{\partial\phi}{\partial y}},\frac{\frac{\partial\phi}{\partial z}}{\frac{\partial\phi}{\partial y}}\right)$$

$$\frac{\partial Z}{\partial(x,y)}=\left(\frac{\partial Z}{\partial x},\frac{\partial Z}{\partial y}\right)=-\left(\frac{\partial F}{\partial Z}\right)^{-1}\frac{\partial F}{\partial(x,y)}$$
$$=-\left(\frac{\frac{\partial\phi}{\partial x}}{\frac{\partial\phi}{\partial z}},\frac{\frac{\partial\phi}{\partial y}}{\frac{\partial\phi}{\partial z}}\right)$$

So
$$\frac{\partial X}{\partial y}\cdot\frac{\partial Y}{\partial z}\cdot\frac{\partial Z}{\partial x}=\left(-\frac{\cancel{\frac{\partial\phi}{\partial y}}}{\cancel{\frac{\partial\phi}{\partial x}}}\right)\left(-\frac{\cancel{\frac{\partial\phi}{\partial z}}}{\cancel{\frac{\partial\phi}{\partial y}}}\right)\left(-\frac{\cancel{\frac{\partial\phi}{\partial x}}}{\cancel{\frac{\partial\phi}{\partial z}}}\right)=(-1)^3=-1.$$