Plenary excercises 0805

10.1.3 $\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$ $\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$

(a)

$$\sin(u)\sin(v) = \frac{e^{iu} - e^{-iu}}{2i} \cdot \frac{e^{iv} - e^{-iv}}{2i} = \frac{e^{i(u+v)} - e^{i(u-v)} - e^{i(v-u)} + e^{i(-u-v)}}{-4}$$

$$= \frac{e^{i(u+v)} + e^{i(-u-v)}}{-4} - \frac{e^{i(u-v)} + e^{i(v-u)}}{-4}$$

$$= -\frac{1}{2}\cos(u+v) + \frac{1}{2}\cos(u-v)$$

(b) $\int \sin 4x \sin x \, dx = \int \frac{1}{2}\cos(3x) - \frac{1}{2}\cos(5x) \, dx$

$$= \frac{1}{2} \cdot \frac{1}{3}\sin(3x) - \frac{1}{2} \cdot \frac{1}{5}\sin(5x) = \frac{1}{6}\sin(3x) - \frac{1}{10}\sin(5x)$$

(c) $\cos(u)\cos(v) = \frac{e^{iu} + e^{-iu}}{2} \cdot \frac{e^{iv} + e^{-iv}}{2} = \frac{e^{i(u+v)} + e^{i(u-v)} + e^{i(v-u)} + e^{i(-u-v)}}{4}$

$$= \frac{e^{i(u+v)} + e^{i(-u-v)}}{4} + \frac{e^{i(u-v)} + e^{i(v-u)}}{4}$$

$$= \frac{\cos(u+v)}{2} + \frac{\cos(u-v)}{2}$$

$$\int \cos(3x)\cos(2x) \, dx = \frac{1}{2}\int \cos(5x) + \cos(x)$$

$$= \frac{1}{2} \cdot \frac{1}{5}\sin(5x) + \frac{1}{2}\sin(x) = \frac{1}{10}\sin(5x) + \frac{1}{2}\sin(x)$$

(d) $\sin(u)\cos(v) = \frac{e^{iu} - e^{-iu}}{2i} \cdot \frac{e^{iv} + e^{-iv}}{2}$

$$= \frac{e^{i(u+v)} + e^{i(u-v)} - e^{i(v-u)} - e^{i(-u-v)}}{4i}$$

$$= \frac{e^{i(u+v)} - e^{i(-u-v)}}{4i} + \frac{e^{i(u-v)} - e^{i(v-u)}}{4i}$$

$$= \frac{1}{2}\sin(u+v) + \frac{1}{2}\sin(u-v)$$

10.1.4

$$\int e^{(a+ib)x}\,dx = \frac{e^{(a+ib)x}}{a+ib} + C$$

$$\frac{e^{(a+ib)x}}{a+ib}\,\frac{a-ib}{a-ib} = \frac{e^{a}\cdot e^{ibx}(a-ib)}{a^2-i^2b^2}$$

$$= \frac{e^{a}\left(\cos(bx)+i\sin(bx)\right)(a-ib)}{a^2+b^2}$$

$$= \frac{e^{a}\left(a\cos(bx)-ib\cos(bx)+ia\sin(bx)+b\sin(bx)\right)}{a^2+b^2}$$

(b) $$\int e^{ax}\cdot e^{ibx}\,dx = \frac{e^{a}\left(a\cos(bx)+b\sin(bx)\right)}{a^2+b^2} + i\,\frac{e^{a}\left(a\sin(bx)-b\cos(bx)\right)}{a^2+b^2}$$

$\downarrow \parallel$

$$\int e^{ax}(\cos bx + i\sin bx) = \int e^{ax}\cos(bx)\,dx + i\int e^{ax}\sin(bx)\,dx$$

Take real and imaginary part on each side we get

$$\int e^{ax}\cos(bx)\,dx = \frac{e^{ax}\left(a\cos(bx)+b\sin(bx)\right)}{a^2+b^2}$$

$$\int e^{ax}\sin(bx)\,dx = \frac{e^{ax}\left(a\sin(bx)-b\cos(bx)\right)}{a^2+b^2}$$

10.1.7 compute Fourier series of f $\sin\left(\frac{x}{2}\right)$

$$\langle f, \alpha_n\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin\left(\frac{x}{2}\right) e^{-inx}\,dx$$

$n=0$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin\left(\frac{x}{2}\right)\cdot dx = \frac{1}{2\pi}\left[-\cos\left(\frac{x}{2}\right)\cdot 2\right]_{-\pi}^{\pi}$$

$$= -\frac{1}{\pi}\left[\cos\left(\frac{x}{2}\right)\right]_{-\pi}^{\pi} = \frac{-1}{\pi}(0-0) = 0$$

$n\neq 0$ Rewrite using excercise 10.1.3

$$\sin\left(\frac{x}{2}\right)e^{-inx} = \sin\left(\frac{x}{2}\right)\cos(-nx) + i\sin\left(\frac{x}{2}\right)\sin(-nx)$$

$$= \sin\left(\frac{x}{2}\right)\cos(nx) - i\sin\left(\frac{x}{2}\right)\sin(nx)$$

$$= \frac{1}{2}\left[\sin\left(\frac{x}{2}+nx\right) + \sin\left(\frac{x}{2}-nx\right)\right]$$

$$-i\frac{1}{2}\left[\cos\left(\frac{x}{2}-nx\right) - \cos\left(\frac{x}{2}+nx\right)\right]$$

want to take integral of

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} dx = \frac{1}{2\pi}\Bigg(\frac{1}{2}\frac{-1}{\frac{1}{2}+n}\cos\left(\frac{x}{2}+nx\right) + \frac{1}{2}\cdot\frac{-1}{\frac{1}{2}-n}\cos\left(\frac{x}{2}-nx\right)$$

$$-i\frac{1}{2}\frac{1}{\frac{1}{2}-n}\sin\left(\frac{x}{2}-nx\right) + \frac{i}{2}\cdot\frac{1}{\frac{1}{2}+n}\sin\left(\frac{x}{2}+nx\right)\Bigg)_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\Bigg[\frac{1}{2}\frac{-1}{\frac{1}{2}+n}\cdot 0 + \frac{1}{2}\cdot\frac{-1}{\frac{1}{2}-n}\cdot 0 - \frac{i}{2}\cdot\frac{1}{\frac{1}{2}-n}(-1)^n + \frac{i}{2}\cdot\frac{1}{\frac{1}{2}+n}(-1)^n$$

$$-\left(\frac{1}{2}\frac{-1}{\frac{1}{2}+n}\cdot 0 + \frac{1}{2}\frac{-1}{\frac{1}{2}-n}\cdot 0 - \frac{i}{2}\frac{1}{\frac{1}{2}-n}(-1)^{n+1} + \frac{i}{2}\frac{1}{\frac{1}{2}+n}(-1)^{n+1}\right)\Bigg]$$

$$= \frac{1}{2\pi}\left[-\frac{i}{2}\cdot\frac{1}{\frac{1}{2}-n}(-1)^n + \frac{i}{2}\frac{1}{\frac{1}{2}+n}(-1)^n + \frac{i}{2}\frac{1}{\frac{1}{2}-n}(-1)^{n+1} - \frac{i}{2}\frac{1}{\frac{1}{2}+n}(-1)^{n+1}\right]$$

$$= \frac{1}{2\pi}\left[-i\cdot\frac{1}{\frac{1}{2}-n}(-1)^n \quad i\frac{1}{\frac{1}{2}+n}(-1)^n\right]$$

$$= \frac{1}{2\pi}(-1)^n\cdot i\left(\frac{1}{\frac{1}{2}+n} - \frac{1}{\frac{1}{2}-n}\right)$$

$$= \frac{i(-1)^n}{2\pi}\left(\frac{2}{1+2n} - \frac{2}{1-2n}\right) = \frac{i(-1)^n}{\pi}\left(\frac{1}{1+2n} - \frac{1}{1-2n}\right)$$

$$= \frac{i(-1)^n}{\pi}\left(\frac{1-2n-(1+2n)}{1-4n^2}\right) = \frac{i(-1)^n}{\pi}\left(\frac{-4n}{1-4n^2}\right)$$

Fourier series becomes

$$\sum_{n=1}^{\infty}\frac{i(-1)^n}{\pi}\left(\frac{-4n}{1-4n^2}\right)\left(\cos(nx) + i\sin(nx)\right)$$

$$+\sum_{n=1}^{\infty}\frac{i(-1)^n}{\pi}\left(\frac{4n}{1-4n^2}\right)\left(\cos(nx) - i\sin(nx)\right)$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^n}{\pi}\left(\frac{4n}{1-4n^2}\right) i\cdot i\left(-\sin(nx) - \sin(nx)\right)$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^n}{\pi}\left(\frac{4n}{1-4n^2}\right) 2\cdot\sin(nx)$$

10.1.8 a) $s_n = a_0 + a_0 r + \ldots + a_0 r^n$

$r s_n = r a_0 + a_0 r^2 + \ldots + a_0 r^{n+1}$

$r s_n - s_n = a_0 r^{n+1} + a_0 r^n + \ldots + a_0 r^2 + a_0 r$
$- a_0 r^n - \ldots - a_0 r^2 - a_0 r - a_0$

$s_n (r-1) = a_0 r^{n+1} - a_0$

If $r \neq 1$

$$s_n = \frac{a_0 (r^{n+1} - 1)}{r-1} \cdot \frac{-1}{-1} = \frac{a_0 (1 - r^{n+1})}{1-r}$$

b) $\sum_{k=0}^{n} e^{ikx}$ geom. rekke med $r = e^{ix}$

$= \frac{1 - e^{i(n+1)x}}{1 - e^{ix}}$ når $e^{ix} \neq 1$

$\Updownarrow$

x er ikke lik $2\pi \cdot l$
der l heltall

c) $\sum_{k=0}^{n} e^{ikx} = e^{i\frac{nx}{2}} \frac{\sin(\frac{n+1}{2}x)}{\sin(\frac{x}{2})}$ vis

$$e^{i\frac{nx}{2}} \frac{\frac{e^{i\frac{n+1}{2}x} - e^{-i\frac{n+1}{2}x}}{2i}}{\frac{e^{i\frac{x}{2}} - e^{-i\frac{x}{2}}}{2i}} = e^{i\frac{nx}{2}} \frac{e^{i\frac{n+1}{2}x} - e^{-i\frac{n+1}{2}x}}{e^{i\frac{x}{2}} - e^{-i\frac{x}{2}}}$$

$$= \frac{e^{i\frac{n+1}{2}x}}{e^{i\frac{x}{2}}} \quad = \frac{e^{i(n+1)x} - 1}{e^{ix} - 1} = \frac{1 - e^{i(n+1)x}}{1 - e^{ix}}$$

$$= \sum_{k=0}^{n} e^{ikx} \text{ ved b.}$$

d) $\sum_{k=0}^{n} e^{ikx} = \sum_{k=0}^{n} \cos(kx) + i\sin(kx) = \frac{\sin(\frac{n+1}{2}x)}{\sin(\frac{x}{2})} \left(\cos\frac{nx}{2} + i\sin\frac{nx}{2}\right)$

Real and imaginary part:

$$\sum_{k=0}^{n} \cos(kx) = \frac{\sin(\frac{n+1}{2}x)}{\sin(\frac{x}{2})} \cos\left(\frac{nx}{2}\right)$$

$$\sum_{k=0}^{n} \sin(kx) = \frac{\sin(\frac{n+1}{2}x)}{\sin(\frac{x}{2})} \sin\left(\frac{nx}{2}\right)$$

10.1.10 f is even $f(-x)=f(x)$
f is odd $f(-x)=-f(x)$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

(a) If a even show $b_n=0$ for all $n=1,2,\ldots$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx=\frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin(nx)\,dx+\frac{1}{\pi}\int_{0}^{\pi} f(x)\sin(nx)\,dx$$

$$\frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin(nx)\,dx \qquad u=-x \qquad \frac{du}{dx}=-1$$

$$du=-dx$$

$$\frac{1}{\pi}\int_{\pi}^{0} f(u)\sin(-nu)\,-du$$

$$=\frac{1}{\pi}-\int_{0}^{\pi} f(u)-\sin(nu)-du=-\frac{1}{\pi}\int_{0}^{\pi} f(u)\sin(nu)\,du$$

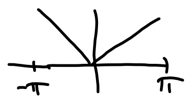
See this is equal to minus $\frac{1}{\pi}\int_{0}^{\pi} f(x)\sin(nx)\,dx$

Thus the sum is zero, so

$$\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx=0$$

f is odd then similar change of variables shows $a_n=0$.

(b) $f(x)=|x|=|-x|=f(-x)$



so f is even

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}|x|=\frac{2}{\pi}\int_{0}^{\pi}|x|\,dx=\frac{2}{\pi}\int_{0}^{\pi}x\,dx=\frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[\frac{\pi^2}{2}-0\right]$$

$$=\pi$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}|x|\cos(nx)\,dx=\frac{2}{\pi}\int_{0}^{\pi}\underset{u}{x}\,\underset{v'}{\cos(nx)}\,dx \qquad u'=1,\ v=\frac{\sin(nx)}{n}$$

$$=\frac{2}{\pi}\left(\frac{x\sin(nx)}{n}\right)_0^{\pi}-\frac{2}{\pi}\int_{0}^{\pi}\frac{\sin(nx)}{n}\,dx$$

$$=\frac{2}{\pi}\Big[\underbrace{\frac{\pi\sin(\pi n)}{n}}_{0}-0\Big]-\left[\frac{2}{\pi}\left(-\frac{\cos(nx)}{n^2}\right)\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[\frac{\cos(nx)}{n^2}\right]_0^{\pi}$$

$$=\frac{2}{\pi}\,\frac{\cos(n\pi)-1}{n^2}=\begin{cases}0 & n \text{ even}\\ -\frac{4}{\pi n^2} & n \text{ odd}\end{cases}$$

Fourier series

$$\frac{\pi}{2}-\frac{4}{\pi}\left(\cos(x)+\frac{\cos(3x)}{3^2}+\frac{\cos(5x)}{5^2}+\cdots\right)$$

10.2.1 $C_p =$ continous $f:[-\pi,\pi] \to \mathbb{C}$
with $f(-\pi)=f(\pi)$.
Will show that C_p is closed in $C([-\pi,\pi],\mathbb{C})$
Will show: given any sequence f_n in C_p which
converges to some f, then f has to be in C_p:
Assume $f_n \to f$ in other words

$$\|f_n - f\| < \varepsilon \quad \text{when } n \geq N$$

$$\sup |f_n(x) - f(x)| < \varepsilon \quad \text{when } n \geq N \quad \text{all } x$$

so $x=\pi$ or $x=-\pi$ $\quad |f_n(\pi)-f(\pi)| < \varepsilon \quad n \geq N$

$$|f_n(-\pi)-f(-\pi)| < \varepsilon \quad n \geq N$$

then $f_n(\pi) \to f(\pi)$

$$f_n(-\pi) \to f(-\pi)$$

Since each $f_n \in C_p$ $\quad f_n(\pi) = f_n(-\pi)$ all n
so the sequences $\{f_n(\pi)\} = \{f_n(-\pi)\}$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ f(\pi) & & f(-\pi) \end{array}$$

So $f(\pi)=f(-\pi)$ by uniqence of convergence.
Then $f \in C_p$ and C_p is closed.

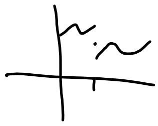
10.2.2 f is in D if $f: [-\pi, \pi] \to \mathbb{C}$
exists finite $a_0 = \pi, a_1 \dots a_n = \pi$

i) f is contenious on (a_i, a_{i+1})

ii) $f(a_i^-) = \lim_{x \to a_i^-} f(x)$

$f(a_i^+) = \lim_{x \to a_i^+} f(x)$ exists

iii) $f(a_i) = \dfrac{f(a_i^+) + f(a_i^-)}{2}$



ⓐ $f, g \in D$ then

f is discont. $a_1 \dots a_n$
g is disont $b_1 \dots b_m$

f+g is continous at all points
i) except $a_1 \dots a_n$ $b_1 \dots b_m$

iii) $\lim_{x \to a_i^-} (f+g)(x) = \lim_{x \to a_i^-} f(x) + g(x) = f(a_i^-) + g(a_i^-)$

iii) $(f+g)(a_i) = f(a_i) + g(a_i) = \dfrac{f(a_i^-) + f(a_i^+)}{2}$

$+ \dfrac{g(a_i^-) + g(a_i^+)}{2}$

$= \dfrac{f(a_i^-) + g(a_i^-) + f(a_i^+) + g(a_i^+)}{2}$

$= \dfrac{(f+g)(a_i^-) + (f+g)(a_i^+)}{2}$

So $f+g \in D$

ⓑ f discont $a_1 \dots a_n$
g is cont

i) fg disont $a_1 \dots a_n$

ii) $fg(a_i^-) = \lim_{x \to a_i^-} f(x)\, g(x) = \lim_{x \to a_i^-} f(x) \cdot g(a_i)$

$= f(a_i^-) g(a_i)$

iii) $fg(a_i) = \dfrac{f(a_i^-) + f(a_i^+)}{2} \cdot g(a_i)$

$= \dfrac{f(a_i^-) g(a_i) + f(a_i^+) g(a_i^+)}{2}$

$= \dfrac{fg(a_i^-) + fg(a_i^+)}{2}$

If f disont $a_1 \dots a_n$
g discont $b_1 \dots b_m$

$a_i = b_j$

$fg(a_i) = \dfrac{f(a_i^-) + f(a_i^+)}{2} \; \dfrac{g(b_j^-) + g(b_j^+)}{2}$

$= \dfrac{f(a_i^-) g(b_j^-) + f(a_i^-) g(b_j^+) + f(a_i^+) g(b_j^-) + f(a_i^+) g(b_j^+)}{4}$

$\neq \dfrac{f(a_i^-) g(b_j^-) + f(a_i^+) g(b_j)^+}{2}$

So $fg \notin D$ even if $f \in D, g \in D$.

10.2.2.c Check the axioms

d) Take $f \in D$.

Define $f_i(x) = \begin{cases} f(x) & x \in (a_i, a_{i+1}) \\ f(a_i^-) & x = a_i \\ f(a_{i+1}^+) & x = a_{i+1} \end{cases}$

then f_i is continuous on $[a_i, a_{i+1}]$

Each f_i is bounded by extreme value thm by M_i. Let $M = \max M_i$

Also let $N = \max f(a_i)$

$K = \max M, N$

Then $f_i(x) \leq K$ on (a_i, a_{i+1}) so

$f(x) \leq k$ on (a_i, a_{i+1})

$f(x) \leq k$ on each a_i by constuction

So $f(x) \leq k$ so $f(x)$ is bounded.

e) $\int_{a_i}^{a_{i+1}} f_i(x)\,dx$ is integrable since it is continous

Also " $\int_{a_i}^{a_{i+1}} f(x)\,dx$ since f and f_i

are different at only two points, since you then can choose identical Rieman sums for both.

Then $\int_{-\pi}^{\pi} f(x)\,dx = \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} f(x)\,dx = \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} f_i(x)\,dx$

The last integral exists thus the First also exists.