**Appendix to**

# Spaces: An Introduction to Real Analysis

## Chapter 10. Fourier Series with Riemann Integration


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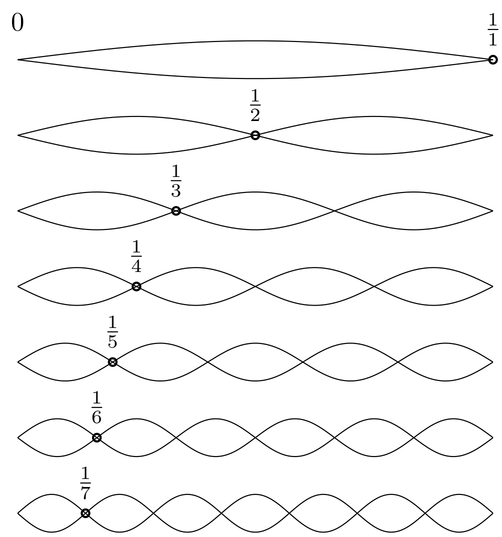
# Fourier Series with Riemann Integration

In the middle of the 18th century, mathematicians and physicists started to study the motion of a vibrating string (think of the strings of a violin or a guitar). If you pull the string out and then let it go, how will it vibrate? To make a mathematical model, assume that at rest the string is stretched along the $x$-axis from 0 to 1 and fastened at both ends.




The figure above shows some possibilities. If we start with a simple sine curve $f_1(x) = C_1 \sin(\pi x)$, the string will oscillate up an down between the two curves shown in the top line of the picture (we are neglecting air resistance and other frictional forces). The frequency of the oscillation is called the *fundamental harmonic* of the string. If we start from a position where the string is pinched at the midpoint as on the second line of the figure (i.e. we use a starting position of the form $f_2(x) = C_2 \sin(2\pi x)$), the string will oscillate with a node in the middle. The frequency will be twice the fundamental harmonic. This is the first overtone of the string. Pinching the string at more and more points, i.e. using starting positions of the form $f_n(x) = C_n \sin(n\pi x)$ for larger and larger integers $n$, we introduce more and more nodes and more and more overtones (the frequency of $f_n$ will be $n$ times the fundamental harmonic). If the string is vibrating in air, the frequencies – the fundamental harmonic and its overtones – can be heard as tones of different pitches.

Imagine now that we start with a mixture

$$f(x) = \sum_{n=1}^{\infty} C_n \sin(n\pi x) \tag{10.0.1}$$

of the starting positions above. The motion of the string will then be a superposition of the motions created by each individual function $f_n(x) = C_n \sin(n\pi x)$. The sound produced will be a mixture of the fundamental harmonic and the different overtones, and the size of the constant $C_n$ will determine how much overtone number $n$ contributes to the sound.

This is a nice description, but the problem is that a function is usually not of the form (10.0.1). Or – perhaps it is? Perhaps any reasonable starting position for the string can be written in the form (10.0.1)? But if so, how do we prove it, and how do we find the coefficients $C_n$? There was a heated discussion on these questions around 1750, but nobody at the time was able to come up with a satisfactory solution.

The solution came with a memoir published by Joseph Fourier in 1807. To understand Fourier's solution, we need to generalize the situation a little. Since the string is fastened at both ends of the interval, a starting position for the string must always satisfy $f(0) = f(1) = 0$. Fourier realized that if he were to include general functions that did not satisfy these boundary conditions in his theory, he needed to allow constant terms and cosine functions in his series. Hence he looked for representations of the form

$$f(x) = A + \sum_{n=1}^{\infty} \big(C_n \sin(n\pi x) + D_n \cos(n\pi x)\big) \tag{10.0.2}$$

with $A, C_n, D_n \in \mathbb{R}$. The big breakthrough was that Fourier managed to find simple formulas to compute the coefficients $A, C_n, D_n$ of this series. This turned trigonometric series into a useful tool in applications (Fourier himself was mainly interested in heat propagation).

When we now begin to develop the theory, we shall change the setting slightly. We shall replace the interval $[0, 1]$ by $[-\pi, \pi]$ (it is easy to go from one interval to another by scaling the functions, and $[-\pi, \pi]$ has certain notational advantages), and we shall replace $\sin(n\pi x)$ and $\cos(n\pi x)$ by complex exponentials $e^{inx}$. Not only does this reduce the types of functions we have to work with from two to one, but it also makes many of our arguments easier and more transparent. We begin by taking a closer look at the relationship between complex exponentials and trigonometric functions.

## 10.1. Fourier coefficients and Fourier series

You may remember the name Fourier from Section 5.3 on inner product spaces, and we shall now see how the abstract Fourier analysis presented there can be turned into concrete Fourier analysis of functions on the real line. Before we do so, it will be convenient to take a brief look at the functions that will serve as elements of our orthonormal basis. Recall that for a complex number $z = x + iy$, the exponential $e^z$ is defined by

$$e^z = e^x(\cos y + i \sin y).$$

We shall mainly be interested in purely imaginary exponents:

$$e^{iy} = \cos y + i \sin y. \tag{10.1.1}$$

Since we also have

$$e^{-iy} = \cos(-y) + i\sin(-y) = \cos y - i \sin y,$$

we may add and subtract to get

$$\cos y = \frac{e^{iy} + e^{-iy}}{2} \tag{10.1.2}$$

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i}. \tag{10.1.3}$$

Formulas (10.1.1)-(10.1.3) give us important connections between complex exponentials and trigonometric functions that we shall exploit later in the section.

We need some information about functions $f : \mathbb{R} \to \mathbb{C}$ of the form

$$f(x) = e^{(a+ib)x} = e^{ax}\cos bx + ie^{ax}\sin bx \qquad \text{where } a \in \mathbb{R}.$$

If we differentiate $f$ by differentiating the real and complex parts separately, we get

$$f'(x) = ae^{ax}\cos bx - be^{ax}\sin bx + iae^{ax}\sin bx + ibe^{ax}\cos bx =$$
$$= ae^{ax}(\cos bx + i\sin bx) + ibe^{ax}(\cos bx + i \sin bx) = (a+ib)e^{(a+ib)x},$$

and hence we have the formula

$$\left(e^{(a+ib)x}\right)' = (a+ib)e^{(a+ib)x} \tag{10.1.4}$$

that we would expect from the real case. Anti-differentiating, we see that

$$\int e^{(a+ib)x}\,dx = \frac{e^{(a+ib)x}}{a+ib} + C, \tag{10.1.5}$$

where $C = C_1 + iC_2$ is an arbitrary, complex constant.

We shall be particularly interested in the functions

$$e_n(x) = e^{inx} = \cos nx + i \sin nx \quad \text{where } n \in \mathbb{Z}.$$

Observe first that these functions are $2\pi$-periodic in the sense that

$$e_n(x + 2\pi) = e^{in(x+2\pi)} = e^{inx}e^{2n\pi i} = e^{inx} \cdot 1 = e_n(x).$$

This means in particular that $e_n(-\pi) = e_n(\pi)$ (they are both equal to $(-1)^n$ as is easily checked). Integrating, we see that for $n \neq 0$, we have

$$\int_{-\pi}^{\pi} e_n(x)\,dx = \left[\frac{e^{inx}}{in}\right]_{-\pi}^{\pi} = \frac{e_n(\pi) - e_n(-\pi)}{in} = 0$$

while we for $n = 0$ have

$$\int_{-\pi}^{\pi} e_0(x)\,dx = \int_{-\pi}^{\pi} 1\,dx = 2\pi.$$

This leads to the following orthogonality relation.

**Proposition 10.1.1.** *For all $n, m \in \mathbb{Z}$ we have (the bar denotes complex conjugation)*

$$\int_{-\pi}^{\pi} e_n(x)\overline{e_m(x)}\,dx = \begin{cases} 0 & \text{if } n \neq m \\ \\ 2\pi & \text{if } n = m \end{cases}$$

**Proof.** Since

$$e_n(x)\overline{e_m(x)} = e^{inx}e^{-imx} = e^{i(n-m)x} = e_{n-m}(x),$$

the lemma follows from the formulas above. □

The proposition shows that the family $\{e_n\}_{n\in\mathbb{Z}}$ is almost orthonormal with respect to the inner product

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx.$$

The only problem is that $\langle e_n, e_n\rangle$ is $2\pi$ and not 1. We could fix this by replacing $e_n$ by $\frac{e_n}{\sqrt{2\pi}}$, but instead we shall choose to change the inner product to

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx.$$

Abusing terminology slightly, we shall refer to this at the $L^2$*-inner product on* $[-\pi, \pi]$. The norm it induces will be called the $L^2$*-norm* $\|\cdot\|_2$. It is defined by

$$\|f\|_2 = \langle f, f\rangle^{\frac{1}{2}} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx\right)^{\frac{1}{2}}.$$

The *Fourier coefficients* of a function $f$ with respect to $\{e_n\}_{n\in\mathbb{Z}}$ are defined by

$$\alpha_n = \langle f, e_n\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{e_n(x)}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx.$$

From Section 5.3 we know that $f = \sum_{n=-\infty}^{\infty} \alpha_n e_n$ (where the series converges in $L^2$-norm) provided $f$ belongs to a space where $\{e_n\}_{n\in\mathbb{Z}}$ is a basis. We shall study this question in detail in the next sections. For the time being, we look at some examples of how to compute the Fourier coefficients $\alpha_n$ and the *Fourier series* $\sum_{n=-\infty}^{\infty} \alpha_n e_n(x)$.

**Example 1:** We shall compute the Fourier coefficients $\alpha_n$ of the function $f(x) = x$. By definition

$$\alpha_n = \langle f, e_n\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}\,dx.$$

It is easy to check that $\alpha_0 = \int_{-\pi}^{\pi} x\,dx = 0$. For $n \neq 0$, we use integration by parts (see Exercise 9) with $u = x$ and $v' = e^{-inx}$. We get $u' = 1$ and $v = \frac{e^{-inx}}{-in}$, and:

$$\alpha_n = -\frac{1}{2\pi}\left[x\frac{e^{-inx}}{in}\right]_{-\pi}^{\pi} + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{-inx}}{in}\,dx =$$

$$= \frac{(-1)^{n+1}}{in} + \frac{1}{2\pi}\left[\frac{e^{-inx}}{n^2}\right]_{-\pi}^{\pi} = \frac{(-1)^{n+1}}{in}.$$

The Fourier series becomes

$$\sum_{n=-\infty}^{\infty}\alpha_n e_n = \sum_{n=-\infty}^{-1}\frac{(-1)^{n+1}}{in}e^{inx} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{in}e^{inx} =$$

$$= \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\cdot\frac{e^{inx}-e^{-inx}}{2i} = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin(nx),$$

where we in the last step have use that $\sin u = \frac{e^{iu}-e^{-iu}}{2i}$. We would like to conclude that $x = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin(nx)$ for $x \in (-\pi, \pi)$, but we don't have the theory to take that step yet. ♣

**A remark on real Fourier series.** Note that in the example above, we started with a real-valued function $f$ and ended up with a series expansion with only real-valued terms. This is a general phenomenon: If the function $f$ is real, we can rewrite its Fourier series as a real series where the functions $e^{inx}$ are replaced by $\cos nx$ and $\sin nx$. The resulting series is called the *real Fourier series of $f$*. Let us take a look at the details.

Assume that $f\colon [-\pi, \pi] \to \mathbb{R}$ is a *real-valued* function with Fourier series $\sum_{n=-\infty}^{\infty}\alpha_n e_n$. Note that since $f$ is real

$$\alpha_{-n} = \frac{1}{2\pi}\int_{\pi}^{\pi} f(x)e^{-i(-nx)}\,dx = \frac{1}{2\pi}\int_{\pi}^{\pi} f(x)\overline{e^{-inx}}\,dx$$

$$= \overline{\frac{1}{2\pi}\int_{\pi}^{\pi} f(x)e^{-inx}\,dx} = \overline{\alpha_n}\,.$$

and hence we can combine the positive and negative terms of the Fourier series in the following way

$$\sum_{n=-\infty}^{\infty}\alpha_n e^{inx} = \alpha_0 + \sum_{n=1}^{\infty}\left(\alpha_n e^{inx} + \alpha_{-n}e^{-inx}\right)$$

$$= \alpha_0 + \sum_{n=1}^{\infty}\left(\alpha_n e^{inx} + \overline{\alpha_n e^{inx}}\right) = \alpha_0 + \sum_{n=1}^{\infty} 2\mathrm{Re}(\alpha_n e^{inx}),$$

where $\mathrm{Re}(z)$ denotes the real part of the complex number $z$. If we put $\alpha_n = c_n + id_n$, we get

$$\mathrm{Re}(\alpha_n e^{inx}) = \mathrm{Re}\big((c_n + id_n)(\cos nx + i\sin nx)\big) = c_n\cos nx - d_n\sin nx,$$

and hence

$$\sum_{n=-\infty}^{\infty}\alpha_n e^{inx} = \alpha_0 + \sum_{n=1}^{\infty}(2c_n\cos nx - 2d_n\sin nx).$$

Let us take a closer look at what $c_n$ and $d_n$ are. We have

$$c_n + id_n = \alpha_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx - i\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx,$$

and since $f$ is real, this implies that $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$ and that $d_n = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$. If we introduce

$$a_n = 2c_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \tag{10.1.6}$$

$$b_n = -2d_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx\ , \tag{10.1.7}$$

we see that we can rewrite the Fourier series of $f$ as

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx).$$

As already mentioned, this is called the *real Fourier series of $f$*.

**Example 2:** Let us compute the real Fourier series of the function

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ \\ 1 & \text{if } x \geq 0 \end{cases}$$

From the symmetry of $f$, we get

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0,$$

and by a similar symmetry argument, we see that

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0$$

for all $n \in \mathbb{N}$ ($f(x)\cos nx$ is an odd function, and hence the contribution to the integral from the interval $[-\pi, 0]$ cancels the contribution from the interval $[0, \pi]$ – see Exercise 10 for more information). Turning to the $b_n$'s, we get

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{0}^{\pi}\sin nx\,dx - \frac{1}{\pi}\int_{-\pi}^{0}\sin nx\,dx$$
$$= \frac{1}{\pi}\left(\left[-\frac{\cos nx}{n}\right]_0^{\pi} - \left[-\frac{\cos nx}{n}\right]_{-\pi}^{0}\right)$$
$$= \frac{1}{\pi}\left(-\frac{\cos n\pi}{n} + 2\frac{\cos 0}{n} - \frac{\cos(-n\pi)}{n}\right)$$
$$= \frac{2}{n\pi}\big(1 - \cos(n\pi)\big) = \begin{cases} \frac{4}{n\pi} & \text{when } n \text{ is odd} \\ \\ 0 & \text{when } n \text{ is even} \end{cases}$$

Hence the real Fourier series of $f$ is

$$\sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi} \sin\big((2k-1)x\big).$$

♣

**Exercises for Section 10.1.**

1. Show that $\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx$ is an inner product on $C([-\pi,\pi],\mathbb{C})$.
2. Deduce the formulas for $\sin(x+y)$ and $\cos(x+y)$ from the rule $e^{i(x+y)} = e^{ix}e^{iy}$.
3. In this problem we shall use complex exponentials to prove some trigonometric identities.
   a) Use the complex expressions for sin and cos to show that
   $$\sin(u)\sin(v) = \frac{1}{2}\cos(u-v) - \frac{1}{2}\cos(u+v).$$
   b) Integrate $\int \sin 4x \sin x\,dx$.
   c) Find a similar expression for $\cos u \cos v$ and use it to compute the integral $\int \cos 3x \cos 2x\,dx$.
   d) Find an expression for $\sin u \cos v$ and use it to integrate $\int \sin x \cos 4x\,dx$.
4. a) Show that if we multiply by the conjugate $a - ib$ in the numerator and the denominator on the right hand side of formula (10.1.5), we get
   $$\int e^{(a+ib)x}\,dx = \frac{e^{ax}}{a^2+b^2}\big(a\cos bx + b\sin bx + i(a\sin bx - b\cos bx)\big).$$
   b) Use the formula in a) to show that
   $$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\big(a\cos bx + b\sin bx\big)$$
   and
   $$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\big(a\sin bx - b\cos bx\big).$$
   In calculus, these formulas are usually proved by two times integration by parts, but in our complex setting they follow more or less immediately from the basic integration formula (10.1.5).
5. Find the Fourier series of $f(x) = e^x$.
6. Find the Fourier series of $f(x) = x^2$.
7. Find the Fourier series of $f(x) = \sin\frac{x}{2}$.
8. a) Let $s_n = a_0 + a_0 r + a_0 r^2 + \cdots + a_0 r^n$ be a geometric series of complex numbers. Show that if $r \neq 1$, then
   $$s_n = \frac{a_0(1-r^{n+1})}{1-r}.$$
   (*Hint:* Subtract $rs_n$ from $s_n$.)
   b) Explain that $\sum_{k=0}^{n} e^{ikx} = \frac{1-e^{i(n+1)x}}{1-e^{ix}}$ when $x$ is not a multiple of $2\pi$.
   c) Show that $\sum_{k=0}^{n} e^{ikx} = e^{i\frac{nx}{2}}\frac{\sin(\frac{n+1}{2}x)}{\sin(\frac{x}{2})}$ when $x$ is not a multiple of $2\pi$.
   d) Use the result in c) to find formulas for $\sum_{k=0}^{n}\cos(kx)$ and $\sum_{k=0}^{n}\sin(kx)$.

9. Show that the integration by parts formula

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

holds for complex-valued functions $f, g$.

10. A real-valued function $f\colon [-\pi,\pi] \to \mathbb{R}$ is called *even* if $f(-x) = f(x)$ for all $x \in [-\pi,\pi]$ and it is called *odd* if $f(-x) = -f(x)$ for all $x \in [-\pi,\pi]$. Let $a_n$ and $b_n$ be the real Fourier coefficients of $f$.

a) Show that if $f$ is even, $b_n = 0$ for all $n = 1, 2, 3, \ldots$, and that if $f$ is odd, $a_n = 0$ for $n = 0, 1, 2, \ldots$. In the first case, we get a *cosine series*

$$\frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos(nx)$$

and in the second case a *sine series*

$$\sum_{n=0}^{\infty} b_n \sin(nx).$$

b) Show that the real Fourier series of $|x|$ is

$$\frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right).$$

c) Show that the real Fourier series of $|\sin x|$ is

$$\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2nx)}{4n^2 - 1}.$$

(*Hint:* Show first that $\sin[(n+1)x] - \sin[(n-1)x] = 2\sin x \cos nx$.)

Let $f\colon [-\pi,\pi] \to \mathbb{R}$ be a real-valued function function with real Fourier series

$$\frac{a_0}{2} + \sum_{n=0}^{\infty} \big(a_n \cos(nx) + b_n \sin(nx)\big).$$

d) Show that $f_e(x) = \frac{f(x)+f(-x)}{2}$ is an even function with real Fourier series

$$\frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos(nx).$$

and that $f_o(x) = \frac{f(x)-f(-x)}{2}$ is an odd function with real Fourier series

$$\sum_{n=0}^{\infty} b_n \sin(nx).$$

11. a) Show that if $b \neq n$, then

$$\int_{-\pi}^{\pi} e^{ibx} \cdot e^{-inx}\,dx = 2(-1)^n \frac{\sin(b\pi)}{b-n}.$$

b) Use a) to find the Fourier series of $\cos(ax)$ when $a \in \mathbb{R}$ *isn't* an integer. What is the Fourier series when $a$ *is* an integer?

12. In this exercise, we shall see how the problem of the vibrating string can be treated by the theory we have started to develop. For simplicity, we assume that the string has length $\pi$ rather than one, and that the initial condition is given by a continuous

function $g\colon [0,\pi] \to \mathbb{R}$ with $g(0) = g(\pi) = 0$. Let $\bar{g}\colon [-\pi,\pi] \to \mathbb{R}$ be the odd extension of $g$, i.e. the function defined by

$$\bar{g}(x) = \begin{cases} g(x) & \text{if } x \in [0,\pi] \\ \\ -g(-x) & \text{if } x \in [-\pi, 0) \end{cases} .$$

a) Explain that the real Fourier series of $\bar{g}$ is a sine series $\sum_{n=1}^{\infty} b_n \sin(nx)$.
b) Show that $b_n = \frac{2}{\pi}\int_0^\pi g(x)\sin(nx)\,dx$.
c) Show that if the sine series converges pointwise to $\bar{g}$, then

$$g(x) = \sum_{n=1}^{\infty} b_n \sin(nx) \qquad \text{for all } x \in [0,\pi].$$

Explain the connection to the vibrating string.

## 10.2. Convergence in mean square

Recall from the previous section that the functions

$$e_n(x) = e^{inx}, \quad n \in \mathbb{Z}$$

form an orthonormal set with respect to the $L^2$-inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx.$$

The Fourier coefficients of a continuous function $f\colon [-\pi,\pi] \to \mathbb{C}$ with respect to this set are given by

$$\alpha_n = \langle f, e_n\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{e_n(x)}\,dx.$$

From Parseval's theorem 5.3.10, we know that if $\{e_n\}$ is a basis (for whatever space we are working with), then

$$f(x) = \sum_{n=-\infty}^{\infty} \alpha_n e_n(x),$$

where the series converges in the $L^2$-norm, i.e.

$$\lim_{N\to\infty} \|f - \sum_{n=-N}^{N} \alpha_n e_n\|_2 = 0.$$

Convergence in $L^2$-norm is also referred to as *convergence in mean square*.

At this stage, life becomes complicated in two ways. First, we don't know yet that $\{e_n\}_{n\in\mathbb{Z}}$ is a basis for $C([-\pi,\pi],\mathbb{C})$, and second, we don't really know what $L^2$-convergence means. It turns out that $L^2$-convergence is quite weak, and that a sequence may converge in $L^2$-norm without actually converging at a single point! This means that we would also like to investigate other forms for convergence (pointwise, uniform etc.).

Let us begin by observing that since $e_n(-\pi) = e_n(\pi)$ for all $n \in \mathbb{Z}$, any function that is the pointwise limit of a series $\sum_{n=-\infty}^{\infty} \alpha_n e_n$ must also satisfy this periodicity assumption. Hence it is natural to introduce the following class of functions:

**Definition 10.2.1.** *Let $C_P$ be the set of all continuous functions $f\colon [-\pi,\pi] \to \mathbb{C}$ such that $f(-\pi) = f(\pi)$. A function in $C_P$ is called a* trigonometric polynomial *if it is of the form $\sum_{n=-N}^{N} \alpha_n e_n$ where $N \in \mathbb{N}$ and each $\alpha_n \in \mathbb{C}$.*

To distinguish it from the $L^2$-norm, we shall denote the supremum norm on $C([-\pi,\pi],\mathbb{C})$ by $\|\cdot\|_\infty$, i.e.

$$\|f\|_\infty = \sup\{|f(x)| \,:\, x \in [-\pi,\pi]\}$$

Note that the metric generated by $\|\cdot\|_\infty$ is the metric $\rho$ that we studied in Chapter 4. Hence convergence with respect to $\|\cdot\|_\infty$ is the same as uniform convergence.

**Theorem 10.2.2.** *The trigonometric polynomials are dense in $C_P$ in the $\|\cdot\|_\infty$-norm. Hence for any $f \in C_P$ there is a sequence $\{p_n\}$ of trigonometric polynomials that converges uniformly to $f$.*

If you have read Section 4.11 on the Stone-Weierstrass Theorem, you may recognize this as Corollary 4.11.13. If you haven't read Section 4.11, don't despair: In the next section, we shall get a more informative proof from ideas we have to develop anyhow, and we postpone the proof till then. In the meantime we look at some consequences.

**Corollary 10.2.3.** *For all $f \in C_P$, the Fourier series $\sum_{n=-\infty}^{\infty} \langle f, e_n\rangle e_n$ converges to $f$ in $L^2$-norm, i.e. $\lim_{N\to\infty} \|f - \sum_{n=-N}^{N} \langle f, e_n\rangle e_n\|_2 = 0$.*

**Proof.** As usual, we let $\alpha_n = \langle f, e_n\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$ be the Fourier coefficients of $f$. Given an $\epsilon > 0$, we must show that there is an $N_0 \in \mathbb{N}$ such that $\|f - \sum_{n=-N}^{N} \alpha_n e_n\|_2 < \epsilon$ for all $N \geq N_0$. By the previous result, we know that there is a trigonometric polynomial $p$ such that $\|f - p\|_\infty < \epsilon$. But then we also have $\|f - p\|_2 < \epsilon$ as

$$\|f-p\|_2 = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t) - p(t)|^2\,dt\right)^{\frac{1}{2}} \leq \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \epsilon^2\,dt\right)^{\frac{1}{2}} = \epsilon.$$

Let $N_0$ be the degree of $p$ and assume that $N \geq N_0$. By Proposition 5.3.8, $\sum_{n=-N}^{N} \alpha_n e_n$ is the element in $\mathrm{Sp}(e_N, e_{-N+1}, \ldots, e_{N-1}, e_N)$ closest to $f$ in $L^2$-norm. As $\mathrm{Sp}(e_N, e_{-N+1}, \ldots, e_{N-1}, e_N)$ consists of the trigonometric polynomials of degree $N$, this means that $\sum_{n=-N}^{N} \alpha_n e_n$ is the trigonometric polynomial of degree $N$ closest to $f$ in $L^2$-norm. Since $p$ is trigonometric polynomial of degree $N$, we get

$$\|f - \sum_{n=-N}^{N} \alpha_n e_n\|_2 \leq \|f - p\|_2 < \epsilon.$$

□

The corollary above is rather unsatisfactory. It is particularly inconvenient that it only applies to periodic functions such that $f(-\pi) = f(\pi)$ (although we can not have *pointwise convergence* to functions violating this condition, we may well have $L^2$-convergence as we soon shall see). To get a better result, we introduce a bigger space $D$ of piecewise continuous functions.

**Definition 10.2.4.** *A function* $f\colon [-\pi,\pi]\to\mathbb{C}$ *is said to be* piecewise continuous with one sided limits *if there exists a finite set of points*

$$-\pi = a_0 < a_1 < a_2 < \ldots < a_{n-1} < a_n = \pi$$

*such that:*

(i) $f$ *is continuous on each interval* $(a_i, a_{i+1})$.

(ii) $f$ *has one-sided limits at each point* $a_i$, *i.e.* $f(a_i^-) = \lim_{x\uparrow a_i} f(x)$ *and* $f(a_i^+) = \lim_{x\downarrow a_i} f(x)$ *both exist, but need not be equal (at the endpoints* $a_0 = -\pi$ *and* $a_n = \pi$ *we do, of course, only require limits from the appropriate side).*

(iii) *The value of* $f$ *at each jump point* $a_i$ *is the average of the one-sided limits, i.e.* $f(a_i) = \frac{1}{2}(f(a_i^-) + f(a_i^+))$. *At the endpoints, this is interpreted as* $f(a_0) = f(a_n) = \frac{1}{2}(f(a_n^-) + f(a_0^+))$.

*The collection of all such functions will be denoted by* $D$.

**Remark:** Part (iii) is only included for technical reasons (we must specify the values at the jump points to make $D$ an inner product space), but it reflects how Fourier series behave — at jump points they always choose the average value. The treatment of the end points may seem particularly strange; why should we enforce the average rule even here? The reason is that since the trigonometric polynomials are $2\pi$-periodic, they regard 0 and $2\pi$ as the "same" point, and hence it is natural to compare the right limit at 0 to the left limit at $2\pi$.

Note that the functions in $D$ are bounded and integrable, that the sum of two functions in $D$ is also in $D$, and that $D$ is an inner product space over $\mathbb{C}$ with the $L^2$-inner product. The next lemma will help us extend Corollary 10.2.3 to $D$.

**Lemma 10.2.5.** $C_P$ *is dense in* $D$ *in the* $L^2$*-norm, i.e. for each* $f \in D$ *and each* $\epsilon > 0$, *there is a* $g \in C_P$ *such that* $\|f - g\|_2 < \epsilon$.

**Proof.** I only sketch the main idea of the proof, leaving the details to the reader. Assume that $f \in D$ and $\epsilon > 0$ are given. To construct $g$, choose a very small $\delta > 0$ (it is your task to figure out how small) and construct $g$ as follows: Outside the (non-overlapping) intervals $(a_i - \delta, a_i + \delta)$, we let $g$ agree with $f$, but in each of these intervals, $g$ follows the straight line connecting the points $(a_i - \delta, f(a_i - \delta))$ and $(a_i + \delta, f(a_i + \delta))$ on $f$'s graph. Check that if we choose $\delta$ small enough, $\|f - g\|_2 < \epsilon$ (In making your choice, you have to take $M = \sup\{|f(x)| : x \in [-\pi,\pi]\}$ into account, and you also have to figure out what to do at the endpoints $-\pi, \pi$ of the interval). □

We can now extend Corollary 10.2.3 above from $C_P$ to $D$.

**Theorem 10.2.6.** *For all* $f \in D$, *the Fourier series* $\sum_{n=-\infty}^{\infty} \langle f, e_n\rangle e_n$ *converges to* $f$ *in* $L^2$*-norm, i.e.* $\lim_{N\to\infty} \|f - \sum_{n=-N}^{N} \langle f, e_n\rangle e_n\|_2 = 0$.

**Proof.** Assume that $f \in D$ and $\epsilon > 0$ are given. By the lemma, we know that there is a $g \in C_P$ such that $\|f - g\|_2 < \frac{\epsilon}{2}$, and by Corollary 10.2.3 there is a trigonometric polynomial $p = \sum_{n=-N}^{N} \alpha_n e_n$ such that $\|g - p\|_2 < \frac{\epsilon}{2}$. The triangle inequality now tells us that

$$\|f - p\|_2 \leq \|f - g\|_2 + \|g - p\|_2 < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Invoking Proposition 5.3.8 again, we see that for $M \geq N$, we have

$$\|f - \sum_{n=-M}^{M} \langle f, e_n \rangle e_n\|_2 \leq \|f - p\|_2 < \epsilon,$$

and the theorem is proved. □

The theorem above is satisfactory in the sense that we know that the Fourier series of $f$ converges to $f$ in $L^2$-norm for a wide class of functions $f$. However, we still have things to attend to: We haven't really proved Theorem 10.2.2 yet, and we would really like to prove that Fourier series converge pointwise (or even uniformly) for a reasonable class of functions. We shall take a closer look at these questions in the next sections.

**Exercises for Section 10.2.**

1. Show that $C_P$ is a closed subset of $C([-\pi, \pi], \mathbb{C})$.
2. In this problem we shall prove some properties of the space $D$.
   a) Show that if $f, g \in D$, then $f + g \in D$.
   b) Show also that if $f \in D$ and $g \in C_P$, then $fg \in D$. Explain that there are functions $f, g \in D$ such that $fg \notin D$.
   c) Show that $D$ is a vector space.
   d) Show that all functions in $D$ are bounded.
   e) Show that all functions in $D$ are integrable on $[-\pi, \pi]$.
   f) Show that $\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx$ is an inner product on $D$.
3. Complete the proof of Lemma 10.2.5.

## 10.3. The Dirichlet kernel

Our arguments so far have been entirely abstract – we have not really used any properties of the functions $e_n(x) = e^{inx}$ except that they are orthonormal and dense in $D$. To get better results, we need to take a closer look at these functions. In some of our arguments, we shall need to change variables in integrals, and such changes may take us outside our basic interval $[-\pi, \pi]$, and hence outside the region where our functions are defined. To avoid these problems, we extend our functions $f \in D$ periodically outside the basic interval such that $f(x + 2\pi) = f(x)$ for all $x \in \mathbb{R}$. Figure 10.3.1 shows the extension graphically; in part a) we have the original function, and in part b) (a part of) the periodic extension. As there is no danger of confusion, we shall denote the original function and the extension by the same symbol $f$.

To see the point of this extension more clearly, assume that we have a function $f\colon [-\pi, \pi] \to \mathbb{R}$. Consider the integral $\int_{-\pi}^{\pi} f(x)\, dx$, and assume that we for some reason want to change variable from $x$ to $u = x + a$. We get

$$\int_{-\pi}^{\pi} f(x)\, dx = \int_{-\pi+a}^{\pi+a} f(u - a)\, du.$$

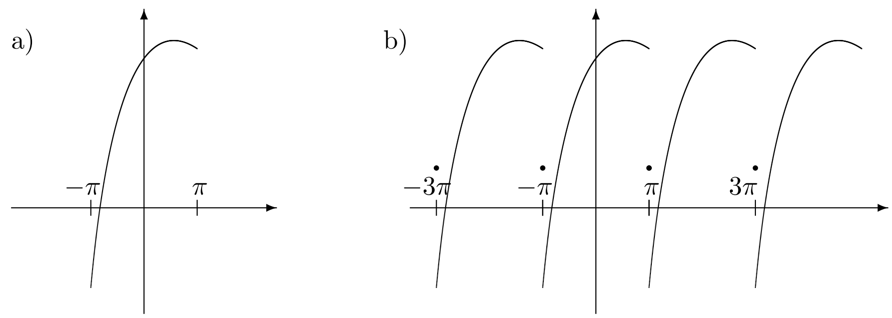



**Figure 10.3.1.** The periodic extension of a function

This is fine, except that we are now longer over our preferred interval $[-\pi, \pi]$. If $f$ has been extended periodically, we see that

$$\int_{\pi}^{\pi+a} f(u-a)\,du = \int_{-\pi}^{-\pi+a} f(u-a)\,du.$$

Hence

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi+a}^{\pi+a} f(u-a)\,du = \int_{-\pi+a}^{\pi} f(u-a)\,du + \int_{\pi}^{\pi+a} f(u-a)\,du$$

$$= \int_{-\pi+a}^{\pi} f(u-a)\,du + \int_{-\pi}^{-\pi+a} f(u-a)\,du = \int_{-\pi}^{\pi} f(u-a)\,du,$$

and we have changed variable without leaving the interval $[-\pi, \pi]$. Variable changes of this sort will be made without further comment in what follows.

**Remark:** Here is a way of thinking that is often useful: Assume that we take our interval $[-\pi, \pi]$ and bend it into a circle such that the points $-\pi$ and $\pi$ become the same. If we think of our trigonometric polynomials $p$ as being defined on the circle instead of on the interval $[-\pi, \pi]$, it becomes quite logical that $p(-\pi) = p(\pi)$. When we are extending functions $f \in D$ the way we did above, we can imagine that we are wrapping the entire real line up around the circle such that the the points $x$ and $x + 2\pi$ on the real line always become the same point on the circle. Mathematicians often say they are "doing Fourier analysis on the unit circle".

Let us begin by looking at the partial sums

$$s_N(x) = \sum_{n=-N}^{N} \langle f, e_n \rangle e_n(x)$$

of the Fourier series. Since

$$\alpha_n = \langle f, e_n \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int}\,dt,$$

we have

$$s_N(x) = \frac{1}{2\pi}\sum_{n=-N}^{N}\left(\int_{-\pi}^{\pi} f(t)e^{-int}\,dt\right)e^{inx} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\sum_{n=-N}^{N} e^{in(x-t)}\,dt =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)\sum_{n=-N}^{N} e^{inu}\,du,$$

where we in the last step have substituted $u = x - t$ and used the periodicity of the functions to remain in the interval $[-\pi, \pi]$. If we introduce the *Dirichlet kernel*

$$D_N(u) = \sum_{n=-N}^{N} e^{inu},$$

we may write this as

$$s_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)D_N(u)\,du.$$

Note that the sum $\sum_{n=-N}^{N} e^{inu} = \sum_{n=-N}^{N}(e^{iu})^n$ in the Dirichlet kernel is a geometric series. For $u = 0$, all the terms are 1 and the sum is $2N + 1$. For $u \neq 0$, we use the summation formula for a finite geometric series to get:

$$D_N(u) = \frac{e^{-iNu} - e^{i(N+1)u}}{1 - e^{iu}} = \frac{e^{-i(N+\frac{1}{2})u} - e^{i(N+\frac{1}{2})u}}{e^{-i\frac{u}{2}} - e^{i\frac{u}{2}}} = \frac{\sin((N+\frac{1}{2})u)}{\sin\frac{u}{2}},$$

where we have used the identity $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$ twice in the last step. This formula gives us a nice, compact expression for $D_N(u)$. If we substitute it into the formula above, we get

$$s_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)\,\frac{\sin((N+\frac{1}{2})u)}{\sin\frac{u}{2}}\,du.$$

If we want to prove that the partial sums $s_N(x)$ converge to $f(x)$ (i.e. that the Fourier series converges pointwise to $f$), the obvious strategy is to prove that the integral above converges to $f$. In 1829, Dirichlet used this approach to prove:

**Theorem 10.3.1** (Dirichlet's Theorem)**.** *If $f \in D$ has only a finite number of local minima and maxima, then the Fourier series of $f$ converges pointwise to $f$.*

Dirichlet's result must have come as something of a surprise; it probably seemed unlikely that a theorem should hold for functions with jumps, but not for continuous functions with an infinite number of extreme points. Through the years that followed, a number of mathematicians tried – and failed – to prove that the Fourier series of a periodic, continuous function always converges pointwise to the function. In 1873, the German mathematician Paul Du Bois-Reymond explained why they failed by constructing a periodic, continuous function whose Fourier series diverges at a dense set of points.

It turns out that the theory for pointwise convergence of Fourier series is quite complicated, and we shall not prove Dirichlet's theorem here. Instead we shall prove a result known as *Dini's test* which allows us to show convergence for many of the functions that appear in practice. But before we do that, we shall take a look at a different notion of convergence which is easier to handle, and which will also give us some tools that are useful in the proof of Dini's test. This alternative notion of convergence is called *Cesaro convergence* or *convergence in Cesaro mean.* However, first of all we shall collect some properties of the Dirichlet kernels that will be useful later.

Let us first see what they look like. Figure 10.3.2 shows Dirichlet's kernel $D_n$ for $n = 5, 10, 15, 20$. Note the changing scale on the $y$-axis; as we have already observed, the maximum value of $D_n$ is $2n + 1$. As $n$ grows, the graph becomes more and more dominated by a sharp peak at the origin. The smaller peaks and valleys shrink in size relative to the big peak, but the problem with the Dirichlet kernel is that they do not shrink in absolute terms — as $n$ goes to infinity, the area between the curve and the $x$-axis (measured in absolute value) goes to infinity. This makes the Dirichlet kernel quite difficult to work with. When we turn to Cesaro convergence in the next section, we get another set of kernels – the *Fejér kernels* – and they turn out not to have this problem. This is the main reason why Cesaro convergence works much better than ordinary convergence for Fourier series.

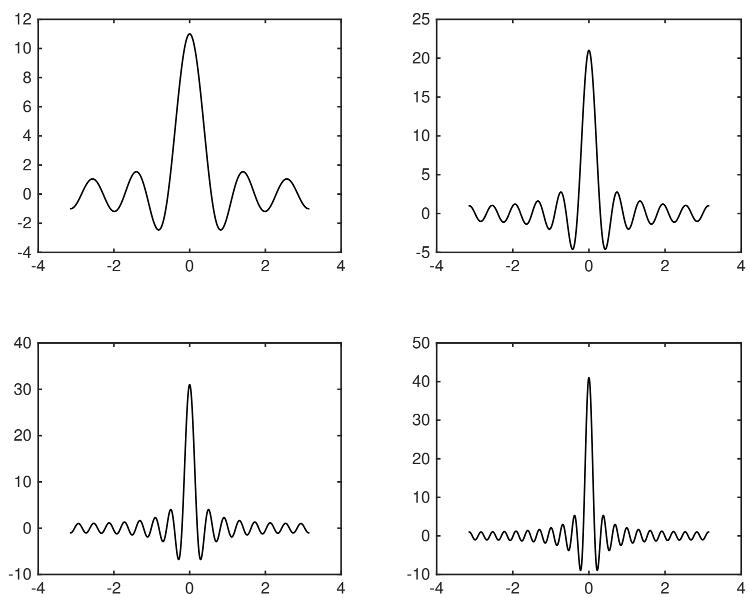



**Figure 10.3.2.** Dirichlet kernels

The following lemma sums up some of the most important properties of the Dirichlet kernel. Recall that a function $g$ is *even* if $g(t) = g(-t)$ for all $t$ in the domain:

**Lemma 10.3.2.** *The Dirichlet kernel $D_n(t)$ is an even, real-valued function such that $|D_n(t)| \leq D_n(0) = 2n+1$ for all $t$. For all $n$,*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(t)\,dt = 1,$$

*but*

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} |D_n(t)|\,dt = \infty.$$

**Proof.** That $D_n$ is real-valued and even, follows immediately from the formula $D_n(t) = \frac{\sin((n+\frac{1}{2})t)}{\sin\frac{t}{2}}$. To prove that $|D_n(t)| \leq D_n(0) = 2n+1$ , we just observe that

$$D_n(t) = |\sum_{k=-n}^{n} e^{ikt}| \leq \sum_{k=-n}^{n} |e^{ikt}| = 2n+1 = D_n(0).$$

Similarly for the integral

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(t)\,dt = \sum_{k=-n}^{n}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikt}\,dt = 1$$

as all integrals except the one for $k=0$ are zero. To prove the last part of the lemma, we observe that since $|\sin u| \leq |u|$ for all $u$, we have

$$|D_n(t)| = \frac{|\sin((n+\frac{1}{2})t)|}{|\sin\frac{t}{2}|} \geq \frac{2|\sin((n+\frac{1}{2})t)|}{|t|}.$$

Using the symmetry and the substitution $z = (n+\frac{1}{2})t$, we see that

$$\int_{-\pi}^{\pi} |D_n(t)|\,dt = \int_{0}^{\pi} 2|D_n(t)|\,dt \geq \int_{0}^{\pi}\frac{4|\sin((n+\frac{1}{2})t)|}{|t|}\,dt =$$

$$= \int_{0}^{(n+\frac{1}{2})\pi}\frac{4|\sin z|}{z}\,dz \geq \sum_{k=1}^{n}\int_{(k-1)\pi}^{k\pi}\frac{4|\sin z|}{k\pi}\,dz = \frac{8}{\pi}\sum_{k=1}^{n}\frac{1}{k}.$$

The expression on the right goes to infinity since the series diverges. □

**Exercises for Section 10.3.**

1. Let $f\colon[-\pi,\pi]\to\mathbb{C}$ be the function $f(x)=x$. Draw the periodic extension of $f$. Do the same with the function $g(x)=x^2$.
2. Check that $D_n(0) = 2n+1$ by computing $\lim_{t\to 0}\frac{\sin((n+\frac{1}{2})t)}{\sin\frac{t}{2}}$.
3. Work out the details of the substitution $u = x-t$ in the derivation of the formula $s_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)\sum_{n=-N}^{N} e^{inu}\,du$.
4. Explain the details in the last part of the proof of Lemma 10.3.2 (the part that proves that $\lim_{n\to\infty}\int_{-\pi}^{\pi}|D_n(t)|\,dt = \infty$).

## 10.4. The Fejér kernel

Before studying the Fejér kernel, we shall take a look at a generalized notion of convergence for sequences. Certain sequences such as

$$0, 1, 0, 1, 0, 1, 0, 1, \ldots$$

do not converge in the ordinary sense, but they do converge "in average" in the sense that the average of the first $n$ elements approaches a limit as $n$ goes to infinity. In this sense, the sequence above obviously converges to $\frac{1}{2}$. Let us make this notion precise:

**Definition 10.4.1.** *Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of complex numbers, and let $S_n = \frac{1}{n}\sum_{k=0}^{n-1} a_k$. We say that the sequence converges to $a \in \mathbb{C}$ in* Cesaro mean *if*

$$a = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a_0 + a_1 + \cdots + a_{n-1}}{n}.$$

*We shall write $a = C\text{-}\lim_{n\to\infty} a_n$.*

The sequence at the beginning of the section converges to $\frac{1}{2}$ in Cesaro mean, but diverges in the ordinary sense. Let us prove that the opposite can not happen:

**Lemma 10.4.2.** *If $\lim_{n\to\infty} a_n = a$, then $C\text{-}\lim_{n\to\infty} a_n = a$.*

**Proof.** Given an $\epsilon > 0$, we must find an $N$ such that

$$|S_n - a| < \epsilon$$

when $n \geq N$. Since $\{a_n\}$ converges to $a$, there is a $K \in \mathbb{N}$ such that $|a_n - a| < \frac{\epsilon}{2}$ when $n \geq K$. If we let $M = \max\{|a_k - a| : k = 0, 1, 2, \ldots\}$, we have for any $n \geq K$:

$$|S_n - a| = \left| \frac{(a_0 - a) + (a_1 - a) + \cdots + (a_{K-1} - a) + (a_K - a) + \cdots (a_{n-1} - a)}{n} \right| \leq$$

$$\leq \left| \frac{(a_0 - a) + (a_1 - a) + \cdots + (a_{K-1} - a)}{n} \right| + \left| \frac{(a_K - a) + \cdots (a_{n-1} - a)}{n} \right| \leq \frac{MK}{n} + \frac{\epsilon}{2}.$$

Choosing $n$ large enough, we get $\frac{MK}{n} < \frac{\epsilon}{2}$, and the lemma follows. □

The idea behind the Fejér kernel is to show that the partial sums $s_n(x)$ converge to $f(x)$ in Cesaro mean; i.e. that the sums

$$S_n(x) = \frac{s_0(x) + s_1(x) + \cdots + s_{n-1}(x)}{n}$$

converge to $f(x)$. Since

$$s_k(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u) D_k(u)\, du,$$

where $D_k$ is the Dirichlet kernel, we get

$$S_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u) \left( \frac{1}{n}\sum_{k=0}^{n-1} D_k(u) \right) du = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u) F_n(u)\, du,$$

where $F_n(u) = \frac{1}{n}\sum_{k=0}^{n-1} D_k(u)$ is the *Fejér kernel*.

We can find a closed expression for the Fejér kernel as we did for the Dirichlet kernel, but the arguments are a little longer:

**Lemma 10.4.3.** *The Fejér kernel is given by*

$$F_n(u) = \frac{\sin^2(\frac{nu}{2})}{n\sin^2(\frac{u}{2})}$$

*for* $u \neq 0$*, and* $F_n(0) = n$*.*

**Proof.** Since

$$F_n(u) = \frac{1}{n}\sum_{k=0}^{n-1} D_k(u) = \frac{1}{n\sin(\frac{u}{2})}\sum_{k=0}^{n-1}\sin((k+\frac{1}{2})u),$$

we have to find

$$\sum_{k=0}^{n-1}\sin((k+\frac{1}{2})u) = \frac{1}{2i}\left(\sum_{k=0}^{n-1} e^{i(k+\frac{1}{2})u} - \sum_{k=0}^{n-1} e^{-i(k+\frac{1}{2})u}\right).$$

The series are geometric and can easily be summed:

$$\sum_{k=0}^{n-1} e^{i(k+\frac{1}{2})u} = e^{i\frac{u}{2}}\sum_{k=0}^{n-1} e^{iku} = e^{i\frac{u}{2}}\frac{1-e^{inu}}{1-e^{iu}} = \frac{1-e^{inu}}{e^{-i\frac{u}{2}} - e^{i\frac{u}{2}}}$$

and

$$\sum_{k=0}^{n-1} e^{-i(k+\frac{1}{2})u} = e^{-i\frac{u}{2}}\sum_{k=0}^{n-1} e^{-iku} = e^{-i\frac{u}{2}}\frac{1-e^{-inu}}{1-e^{-iu}} = \frac{1-e^{-inu}}{e^{i\frac{u}{2}} - e^{-i\frac{u}{2}}}.$$

Hence

$$\sum_{k=0}^{n-1}\sin((k+\frac{1}{2})u) = \frac{1}{2i}\left(\frac{1-e^{inu}+1-e^{-inu}}{e^{-i\frac{u}{2}} - e^{i\frac{u}{2}}}\right) = \frac{1}{2i}\left(\frac{e^{inu}-2+e^{-inu}}{e^{i\frac{u}{2}} - e^{-i\frac{u}{2}}}\right) =$$

$$= \frac{1}{2i}\cdot\frac{(e^{i\frac{nu}{2}} - e^{-\frac{nu}{2}})^2}{e^{i\frac{u}{2}} - e^{-i\frac{u}{2}}} = \frac{\left(\frac{e^{i\frac{nu}{2}} - e^{-\frac{nu}{2}})}{2i}\right)^2}{\frac{e^{i\frac{u}{2}} - e^{-i\frac{u}{2}}}{2i}} = \frac{\sin^2(\frac{nu}{2})}{\sin\frac{u}{2}},$$

and thus

$$F_n(u) = \frac{1}{n\sin(\frac{u}{2})}\sum_{k=0}^{n-1}\sin((k+\frac{1}{2})u) = \frac{\sin^2(\frac{nu}{2})}{n\sin^2\frac{u}{2}}.$$

To prove that $F_n(0) = n$, we just have to sum an arithmetic series

$$F_n(0) = \frac{1}{n}\sum_{k=0}^{n-1} D_k(0) = \frac{1}{n}\sum_{k=0}^{n-1}(2k+1) = n.$$

□

Figure 10.4.1 shows the Fejér kernels $F_n$ for $n = 5, 10, 15, 20$. At first glance they look very much like the Dirichlet kernels in the previous section. The peak in the middle is growing slower than before in absolute terms (the maximum value is $n$ compared to $2n+1$ for the Dirichlet kernel), but relative to the smaller peaks and valleys, it is much more dominant. The functions are now positive, and the area between their graphs and the $x$-axis is always equal to one. As $n$ gets big, almost all this area belongs to the dominant peak in the middle. The positivity and the concentration of all the area in the center peak make the Fejér kernels much easier to handle than their Dirichlet counterparts.

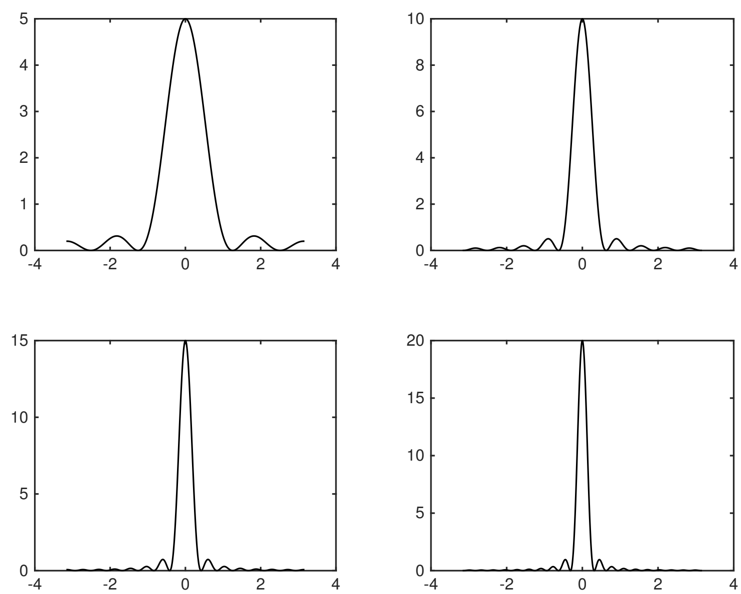



**Figure 10.4.1.** Fejér kernels

Let us now prove some basic properties of the Fejér kernels.

**Proposition 10.4.4.** *For all $n$, the Fejér kernel $F_n$ is an even, positive function such that*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F_n(x)\,dx = 1.$$

*For all nonzero $x \in [-\pi, \pi]$*

$$0 \le F_n(x) \le \frac{\pi^2}{nx^2}.$$

**Proof.** That $F_n$ is even and positive follows directly from the formula in the lemma. By Proposition 10.3.2, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F_n(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{n}\sum_{k=0}^{n-1} D_k\,dx = \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{2\pi}\int_{-\pi}^{\pi} D_k\,dx = \frac{1}{n}\sum_{k=0}^{n-1} 1 = 1.$$

For the last formula, observe that for $u \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, we have $\frac{2}{\pi}|u| \le |\sin u|$ (make a drawing). Thus

$$F_n(x) = \frac{\sin^2(\frac{nx}{2})}{n\sin^2\frac{x}{2}} \le \frac{1}{n(\frac{2}{\pi}\frac{x}{2})^2} \le \frac{\pi^2}{nx^2}.$$

□

We shall now show that if $f \in D$, then $S_n(x)$ converges to $f(x)$, i.e. that the Fourier series converges to $f$ in Cesaro mean. We have already observed that

$$S_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)F_n(u)\,du.$$

If we introduce a new variable $t = -u$ and use that $F_n$ is even, we get

$$S_n(x) = \frac{1}{2\pi}\int_{\pi}^{-\pi} f(x+t)F_n(-t)\,(-dt) =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t)F_n(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+u)F_n(u)\,du.$$

If we take the average of the two expressions we now have for $S_n(x)$, we get

$$S_n(x) = \frac{1}{4\pi}\int_{-\pi}^{\pi} \left(f(x+u) + f(x-u)\right) F_n(u)\,du.$$

Since $\frac{1}{2\pi}\int_{-\pi}^{\pi} F_n(u)\,du = 1$, we also have

$$f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)F_n(u)\,du.$$

Hence

$$S_n(x) - f(x) = \frac{1}{4\pi}\int_{-\pi}^{\pi} \big(f(x+u) + f(x-u) - 2f(x)\big)F_n(u)\,du.$$

To prove that $S_n(x)$ converges to $f(x)$, we only need to prove that the integral goes to 0 as $n$ goes to infinity. The intuitive reason for this is that for large $n$, the kernel $F_n(u)$ is extremely small except when $u$ is close to 0, but when $u$ is close to 0, the other factor in the integral, $f(x+u)+f(x-u)-2f(x)$, is very small (see the proof below for details).

**Theorem 10.4.5** (Fejér's Theorem)**.** *If $f \in D$, then $S_n$ converges to $f$ on $[-\pi, \pi]$, i.e. the Fourier series converges in Cesaro mean. The convergence is uniform on each subinterval $[a, b] \subseteq [-\pi, \pi]$ where $f$ is continuous.*

**Proof.** Given $\epsilon > 0$, we must find an $N \in \mathbb{N}$ such that $|S_n(x) - f(x)| < \epsilon$ when $n \geq N$. Since $f$ is in $D$, there is a $\delta > 0$ such that

$$|f(x+u) + f(x-u) - 2f(x)| < \epsilon$$

when $|u| < \delta$ (keep in mind that since $f \in D$, $f(x) = \frac{1}{2}\lim_{u\uparrow 0}[f(x+u) - f(x-u)]$). We have

$$|S_n(x) - f(x)| \leq \frac{1}{4\pi}\int_{-\pi}^{\pi} |f(x+u) + f(x-u) - 2f(x)|\,F_n(u)\,du =$$

$$= \frac{1}{4\pi}\int_{-\delta}^{\delta} |f(x+u) + f(x-u) - 2f(x)|\,F_n(u)\,du +$$

$$+ \frac{1}{4\pi}\int_{-\pi}^{-\delta} |f(x+u) + f(x-u) - 2f(x)|\,F_n(u)\,du +$$

$$+ \frac{1}{4\pi}\int_{\delta}^{\pi} |f(x+u) + f(x-u) - 2f(x)|\,F_n(u)\,du.$$

For the first integral we have

$$\frac{1}{4\pi}\int_{-\delta}^{\delta}|f(x+u)+f(x-u)-2f(x)|\,F_n(u)\,du \le$$

$$\le \frac{1}{4\pi}\int_{-\delta}^{\delta}\epsilon F_n(u)\,du \le \frac{1}{4\pi}\int_{-\pi}^{\pi}\epsilon F_n(u)\,du = \frac{\epsilon}{2}.$$

For the second integral we get (using the second part of Proposition 10.4.4)

$$\frac{1}{4\pi}\int_{-\pi}^{-\delta}|f(x+u)+f(x-u)-2f(x)|\,F_n(u)\,du \le$$

$$\le \frac{1}{4\pi}\int_{-\pi}^{-\delta}4\|f\|_\infty\frac{\pi^2}{n\delta^2}\,du = \frac{\pi^2\|f\|_\infty}{n\delta^2}.$$

Exactly the same estimate holds for the third integral, and by choosing $N > \frac{4\pi^2\|f\|_\infty}{\epsilon\delta^2}$, we get the sum of the last two integrals less than $\frac{\epsilon}{2}$. But then $|S_n(x) - f(x)| < \epsilon$ and the convergence is proved.

So what about the uniform convergence? We need to check that we can choose the same $N$ for all $x \in [a,b]$. Note that $N$ only depends on $x$ through the choice of $\delta$, and hence it suffices to show that we can use the same $\delta$ for all $x \in [a,b]$.

Since $f \in D$, it has to be continuous on an interval $[a-\eta, b+\eta]$ slightly larger than $[a,b]$, and since $[a-\eta, b+\eta]$ is compact, $f$ is uniformly continuous on $[a-\eta, b+\eta]$. Hence there is a $\delta$, $0 < \delta \le \eta$, such that if $|u| < \delta$, then

$$|f(x+u)+f(x-u)-2f(x)| \le |f(x+u)-f(x)| + |f(x-u)-f(x)| < \epsilon$$

for all $x \in [a,b]$. This proves that we can choose the same $\delta$ for all $x \in [a,b]$, and as already observed the uniform convergence follows. □

We have now finally proved Theorem 10.2.2 which we restate here:

**Corollary 10.4.6.** *The trigonometric polynomials are dense in $C_P$ in $\|\cdot\|_\infty$-norm, i.e. for any $f \in C_P$ there is a sequence of trigonometric polynomials converging uniformly to $f$.*

**Proof.** According to the theorem, the sums $S_N(x) = \frac{1}{N}\sum_{n=0}^{N-1} s_n(x)$ converge uniformly to $f$. Since each $s_n$ is a trigonometric polynomial, so are the $S_N$'s. □

**Exercises to Section 10.4.**

1. Let $\{a_n\}$ be the sequence $1,0,1,0,1,0,1,0,\ldots$. Prove that $\text{C-}\lim_{n\to\infty} a_n = \frac{1}{2}$.
2. Assume that $\{a_n\}$ and $\{b_n\}$ converge in Cesaro mean. Show that
$$\text{C-}\lim_{n\to\infty}(a_n+b_n) = \text{C-}\lim_{n\to\infty} a_n + \text{C-}\lim_{n\to\infty} b_n$$
3. Check that $F_n(0) = n$ by computing $\lim_{u\to 0}\frac{\sin^2(\frac{nu}{2})}{n\sin^2\frac{u}{2}}$.
4. Show that $S_N(x) = \sum_{n=-(N-1)}^{N-1}\alpha_n(1-\frac{|n|}{N})e_n(x)$, where $\alpha_n = \langle f, e_n\rangle$ is the Fourier coefficient.
5. Assume that $f \in C_P$. Work through the details of the proof of Theorem 10.4.5 and check that $S_n$ converges uniformly to $f$.

6. Assume that for each $n \in \mathbb{N}$, $K_n : [-\pi, \pi] \to \mathbb{R}$ is a continuous function. We say that $\{K_n\}$ is a sequence of *good kernels* if the following conditions are satisfied:
   (i) For all $n \in \mathbb{N}$, $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)\,dx = 1$.
   (ii) There is an $M$ such that $\int_{-\pi}^{\pi} |K_n(x)|\,dx \le M$ for all $n \in \mathbb{N}$.
   (iii) For every $\delta > 0$, we have $\lim_{n\to\infty}\int_{-\pi}^{-\delta} |K_n(x)|\,dx = \lim_{n\to\infty}\int_{\delta}^{\pi} |K_n(x)|\,dx = 0$.
   a) Show that the Fejér kernels $\{F_n\}$ form a sequence of good kernels while the Dirichlet kernels $\{D_n\}$ do not.
   b) Assume that $\{K_n\}$ is a sequence of good kernels. For $f \in C_P$, let

$$s_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)K_n(u)\,du.$$

   Show that $\{s_n\}$ converges uniformly to $f$.

## 10.5. The Riemann-Lebesgue lemma

The Riemann-Lebesgue lemma is a seemingly simple observation about the size of the Fourier coefficients, but it turns out to be a very efficient tool in the study of pointwise convergence.

**Theorem 10.5.1** (Riemann-Lebesgue Lemma)**.** *If $f \in D$ and*

$$\alpha_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx, \quad n \in \mathbb{Z},$$

*are the Fourier coefficients of $f$, then* $\lim_{|n|\to\infty} \alpha_n \to 0$.

**Proof.** According to Bessel's inequality 5.3.9, $\sum_{n=-\infty}^{\infty} |\alpha_n|^2 \le \|f\|_2^2 < \infty$, and hence $\alpha_n \to 0$ as $|n| \to \infty$. □

**Remark:** We are cheating a little here as we only prove the Riemann-Lebesgue lemma for function which are in $D$ and hence square integrable. The lemma holds for integrable functions in general, but even in that case the proof is quite easy.

The Riemann-Lebesgue lemma is rather deceptive. It seems to be a result about the coefficients of certain series, and it is proved by very general and abstract methods, but it is really a theorem about oscillating integrals as the following corollary makes clear.

**Corollary 10.5.2.** *If $f \in D$ and $[a, b] \subseteq [-\pi, \pi]$, then*

$$\lim_{|n|\to\infty}\int_a^b f(x)e^{-inx}\,dx = 0.$$

*Also*

$$\lim_{|n|\to\infty}\int_a^b f(x)\cos(nx)\,dx = \lim_{|n|\to\infty}\int_a^b f(x)\sin(nx)\,dx = 0.$$

**Proof.** Let $g$ be the function (this looks more horrible than it is!)

$$g(x) = \begin{cases} 0 & \text{if } x \notin [a,b] \\ f(x) & \text{if } x \in (a,b) \\ \frac{1}{2}\lim_{x\downarrow a} f(x) & \text{if } x = a \\ \frac{1}{2}\lim_{x\uparrow b} f(x) & \text{if } x = b \end{cases}$$

Then $g$ is in $D$, and

$$\int_a^b f(x)e^{-inx}\,dx = \int_{-\pi}^{\pi} g(x)e^{-inx}\,dx = 2\pi\alpha_n,$$

where $\alpha_n$ is the Fourier coefficient of $g$. By the Riemann-Lebesgue lemma, $\alpha_n \to 0$. The last two parts follows from what we have just proved and the identities $\sin(nx) = \frac{e^{inx}-e^{-inx}}{2i}$ and $\cos(nx) = \frac{e^{inx}+e^{-inx}}{2}$ □

Let us pause for a moment to discuss why these results hold. The reason is simply that for large values of $n$, the functions $\sin nx$, $\cos nx$, and $e^{inx}$ (if we consider the real and imaginary parts separately) oscillate between positive and negative values. If the function $f$ is relatively smooth, the positive and negative contributions cancel more and more as $n$ increases, and in the limit there is nothing left. This argument also indicates why rapidly oscillating, continuous functions are a bigger challenge for Fourier analysis than jump discontinuities – functions with jumps average out on each side of the jump, while for wildly oscillating functions "the averaging" procedure may not work.

Since the Dirichlet kernel contains the factor $\sin((n+\frac{1}{2})x)$, the following result will be useful in the next section:

**Corollary 10.5.3.** *If $f \in D$ and $[a,b] \subseteq [-\pi,\pi]$, then*

$$\lim_{|n|\to\infty} \int_a^b f(x)\sin\big((n+\frac{1}{2})x\big)\,dx = 0.$$

**Proof.** Follows from the corollary above and the identity

$$\sin\big((n+\frac{1}{2})x\big) = \sin(nx)\cos\frac{x}{2} + \cos(nx)\sin\frac{x}{2}.$$

□

**Exercises to Section 10.5.**

1. Work out the details of the $\sin(nx)$- and $\cos(nx)$-part of Corollary 10.5.2.
2. Work out the details of the proof of Corollary 10.5.3.
3. a) Show that if $p$ is a trigonometric polynomial, then the Fourier coefficients $\beta_n = \langle p, e_n\rangle$ are zero when $|n|$ is sufficiently large.
   b) Let $f$ be an integrable function, and assume that for each $\epsilon > 0$ there is a trigonometric polynomial such that $\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t) - p(t)|\,dt < \epsilon$. Show that if $\alpha_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt$ are the Fourier coefficients of $f$, then $\lim_{|n|\to\infty} \alpha_n = 0$.

4. If $f, g\colon \mathbb{R} \to \mathbb{R}$ are two continuous, $2\pi$-periodic functions (i.e. $f(x+2\pi) = f(x)$ and $g(x+2\pi) = g(x)$ for all $x \in \mathbb{R}$), we define the *convolution* $f * g$ to be the function
$$(f*g)(u) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u-x)g(x)\,dx$$
   a) Show that $f * g = g * f$.
   b) Show that if
$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx \quad \text{and} \quad b_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(y)e^{-iny}\,dy$$
   are the Fourier coefficients of $f$ and $g$, and
$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}(f*g)(u)e^{-inu}\,du$$
   is the Fourier coefficient of $f * g$, then $c_n = a_n b_n$.
   c) Show that there is no continuous, $2\pi$-periodic function $k\colon \mathbb{R} \to \mathbb{R}$ such that $k * f = f$ for all continuous $f$.
5. We shall prove the following statement:

   *Assume that $f \in D$ has Fourier coefficients $\alpha_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$. If there are positive constants $c, \gamma \in \mathbb{R}_+$ such that*
$$|f(x) - f(y)| \leq c|x-y|^{\gamma}$$
   *for all $x, y \in [-\pi, \pi]$, then*
$$|\alpha_n| \leq \frac{c}{2}\left(\frac{\pi}{n}\right)^{\gamma}$$
   *for all $n \in \mathbb{Z}$.*

   Explain the following calculations and show that they prove the statement:
$$\alpha_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi-\frac{\pi}{n}}^{\pi-\frac{\pi}{n}} f(t+\frac{\pi}{n})e^{-in(t+\frac{\pi}{n})}\,dt$$
$$= -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t+\frac{\pi}{n})e^{-int}\,dt = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+\frac{\pi}{n})e^{-inx}\,dx.$$
   Hence
$$|\alpha_n| = |\frac{1}{4\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx - \frac{1}{4\pi}\int_{-\pi}^{\pi} f(x+\frac{\pi}{n})e^{-inx}\,dx|$$
$$\leq \frac{1}{4\pi}\int_{-\pi}^{\pi}|f(x) - f(x+\frac{\pi}{n})|\,dx \leq \frac{1}{4\pi}\int_{-\pi}^{\pi} c\left(\frac{\pi}{n}\right)^{\gamma}\,dx = \frac{c}{2}\left(\frac{\pi}{n}\right)^{\gamma}.$$
   *Remark:* This result connects the "smoothness" of $f$ (the larger $\gamma$ is, the smoother $f$ is) to the decay of the Fourier coefficients: Roughly speaking, the smoother the function is, the faster the Fourier coefficients decay (recall that by the Riemann-Lebesgue Lemma, $|\alpha_n| \to 0$). This is an important theme in Fourier analysis.

## 10.6. Dini's Test

We shall finally take a serious look at pointwise convergence of Fourier series. As already indicated, this is a rather tricky business, and there is no ultimate theorem, just a collection of scattered results useful in different settings. We shall concentrate on a criterion called *Dini's test* which is relatively easy to prove and sufficiently general to cover a lot of different situations.

Recall from Section 10.3 that if

$$s_N(x) = \sum_{n=-N}^{N} \langle f, e_n \rangle e_n(x)$$

is the partial sum of a Fourier series, then

$$s_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-u) D_N(u)\, du.$$

If we change variable in the integral and use the symmetry of $D_N$, we see that we also have

$$s_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+u) D_N(u)\, du.$$

Taking the average of these two expressions, we get

$$s_N(x) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \big(f(x+u) + f(x-u)\big) D_N(u)\, du.$$

Since $\frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(u)\, du = 1$, we also have

$$f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) D_N(u)\, du,$$

and hence

$$s_N(x) - f(x) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \big(f(x+u) + f(x-u) - 2f(x)\big) D_N(u)\, du$$

(note that the we are now doing exactly the same to the Dirichlet kernel as we did to the Fejér kernel in Section 10.4). To prove that the Fourier series converges pointwise to $f$, we just have to prove that the integral converges to 0.

The next lemma simplifies the problem by telling us that we can concentrate on what happens close to the origin:

**Lemma 10.6.1.** *Assume that $f \in D$. Let $x \in [-\pi, \pi]$, and assume that there is a $\eta > 0$ such that*

$$\lim_{N\to\infty} \frac{1}{4\pi} \int_{-\eta}^{\eta} \big(f(x+u) + f(x-u) - 2f(x)\big) D_N(u)\, du = 0.$$

*Then the Fourier series converges to the function $f$ at the point $x$, i.e. $s_N(x) \to f(x)$*

**Proof.** Note that since $\frac{1}{\sin\frac{x}{2}}$ is a bounded function on $[\eta, \pi]$, Corollary 10.5.3 tells us that

$$\lim_{N\to\infty} \frac{1}{4\pi} \int_{\eta}^{\pi} \big(f(x+u) + f(x-u) - 2f(x)\big) D_N(u)\, du =$$

$$= \lim_{N\to\infty} \frac{1}{4\pi} \int_{\eta}^{\pi} \Big[\big(f(x+u) + f(x-u) - 2f(x)\big) \frac{1}{\sin\frac{u}{2}}\Big] \sin\big((N+\frac{1}{2})u\big)\, du = 0.$$

The same obviously holds for the integral from $-\pi$ to $-\eta$, and hence

$$s_N(x) - f(x) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \big(f(x+u) + f(x-u) - 2f(x)\big) D_N(u)\, du =$$

$$= \frac{1}{4\pi}\int_{-\pi}^{\eta} \big(f(x+u)+f(x-u)-2f(x)\big)D_N(u)\,du +$$
$$+\frac{1}{4\pi}\int_{-\eta}^{\eta} \big(f(x+u)+f(x-u)-2f(x)\big)D_N(u)\,du +$$
$$+\frac{1}{4\pi}\int_{\eta}^{\pi} \big(f(x+u)+f(x-u)-2f(x)\big)D_N(u)\,du$$
$$\to 0+0+0=0.$$

□

**Theorem 10.6.2** (Dini's Test)**.** *Assume that $f \in D$. Let $x \in [-\pi,\pi]$, and assume that there is a $\delta > 0$ such that*

$$\int_{-\delta}^{\delta} \left| \frac{f(x+u)+f(x-u)-2f(x)}{u} \right| du < \infty.$$

*Then the Fourier series converges to the function $f$ at the point $x$, i.e. $s_N(x) \to f(x)$.*

**Proof.** According to the lemma, it suffices to prove that

$$\lim_{N\to\infty} \frac{1}{4\pi}\int_{-\delta}^{\delta} (f(x+u)+f(x-u)-2f(x))\,D_N(u)\,du = 0.$$

Given an $\epsilon > 0$, we have to show that if $N \in \mathbb{N}$ is large enough, then

$$\frac{1}{4\pi}\int_{-\delta}^{\delta} \big(f(x+u)+f(x-u)-2f(x)\big)D_N(u)\,du < \epsilon.$$

Since the integral in the theorem converges, there is an $\eta > 0$ such that

$$\int_{-\eta}^{\eta} \left| \frac{f(x+u)+f(x-u)-2f(x)}{u} \right| du < \epsilon.$$

Since $|\sin v| \geq \frac{2|v|}{\pi}$ for $v \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ (make a drawing), we have $|D_N(u)| = |\frac{\sin((N+\frac{1}{2})u)}{\sin\frac{u}{2}}| \leq \frac{\pi}{|u|}$ for $u \in [-\pi,\pi]$. Hence

$$|\frac{1}{4\pi}\int_{-\eta}^{\eta} \big(f(x+u)+f(x-u)-2f(x)\big)D_N(u)\,du| \leq$$
$$\leq \frac{1}{4\pi}\int_{-\eta}^{\eta} |f(x+u)+f(x-u)-2f(x)|\,\frac{\pi}{|u|}\,du < \frac{\epsilon}{4}.$$

Using Corollary 10.5.3 the same way as in the previous proof, we see that we can get

$$\frac{1}{4\pi}\int_{\eta}^{\delta} \big(f(x+u)+f(x-u)-2f(x)\big)D_N(u)\,du$$

as small as we want by choosing $N$ large enough, and similarly for the integral from $-\delta$ to $-\eta$. In particular, we can get

$$\frac{1}{4\pi}\int_{-\delta}^{\delta} \big(f(x+u)+f(x-u)-2f(x)\big)D_N(u)\,du =$$
$$= \frac{1}{4\pi}\int_{-\delta}^{-\eta} \big(f(x+u)+f(x-u)-2f(x)\big)D_N(u)\,du +$$

$$+\frac{1}{4\pi}\int_{-\eta}^{\eta}\big(f(x+u)+f(x-u)-2f(x)\big)D_N(u)\,du+$$
$$+\frac{1}{4\pi}\int_{\eta}^{\delta}\big(f(x+u)+f(x-u)-2f(x)\big)D_N(u)\,du$$

less than $\epsilon$, and hence the theorem is proved. □

Dini's test has some immediate consequences that we leave to the reader to prove.

**Corollary 10.6.3.** *If $f \in D$ is differentiable at a point $x$, then the Fourier series converges to $f(x)$ at this point.*

We may even extend this result to one-sided derivatives:

**Corollary 10.6.4.** *Assume $f \in D$ and that the limits*

$$\lim_{u\downarrow 0}\frac{f(x+u)-f(x^+)}{u}$$

*and*

$$\lim_{u\uparrow 0}\frac{f(x+u)-f(x^-)}{u}$$

*exist at a point $x$. Then the Fourier series $s_N(x)$ converges to $f(x)$ at this point.*

**Exercises to Section 10.6.**

1. Show that the Fourier series $\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin(nx)$ in Example 10.1.1 converges to $f(x)=x$ for $x \in (-\pi,\pi)$. What happens at the endpoints?
2. In Example 2 in Section 10.1 we showed that the real Fourier series of the function
$$f(x)=\begin{cases}-1 & \text{if } x<0\\ \\ 1 & \text{if } x\geq 0\end{cases}$$
is $\sum_{n=1}^{\infty}\frac{4}{(2k-1)\pi}\sin\big((2k-1)x\big)$. Describe the limit of the series for all $x \in \mathbb{R}$.
3. Prove Corollary 10.6.3
4. Prove Corollary 10.6.4
5. Show that if $a \in \mathbb{R}$, $a \neq 0$, then
$$e^{ax}=\frac{e^{a\pi}-e^{-a\pi}}{\pi}\left(\frac{1}{2a}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+a^2}\big(a\cos nx-n\sin nx\big)\right)$$
for all $x \in (-\pi,\pi)$.
6. Show that for $x \in (0,2\pi)$,
$$x=\pi-2\left(\sin x+\frac{\sin 2x}{2}+\frac{\sin 3x}{3}+\dots\right).$$
(*Warning:* Note that the interval is not the usual $[-\pi,\pi]$. This has to be taken into account.)
7. Let the function $f$ be defined on $[-\pi,\pi]$ by
$$f(x)=\begin{cases}\frac{\sin x}{x} & \text{for } x\neq 0\\ \\ 1 & \text{for } x=0\end{cases}$$
and extend $f$ periodically to all of $\mathbb{R}$.

a) Show that
$$f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$$
where
$$c_n = \frac{1}{2\pi} \int_{(n-1)\pi}^{(n+1)\pi} \frac{\sin x}{x}\, dx.$$
(*Hint:* Write $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$ and use the changes of variable $z = (n+1)x$ and $z = (n-1)x$.)

b) Use this to compute the integral
$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx.$$

8. Let $0 < r < 1$ and consider the series
$$\sum_{-\infty}^{\infty} r^{|n|} e^{inx}.$$

a) Show that the series converges uniformly on $\mathbb{R}$, and that the sum equals
$$P_r(x) = \frac{1 - r^2}{1 - 2r\cos x + r^2}\ .$$
This expression is known as the *Poisson kernel*.

b) Show that $P_r(x) \geq 0$ for all $x \in \mathbb{R}$.

c) Show that for every $\delta \in (0, \pi)$, $P_r(x)$ converges uniformly to 0 on the intervals $[-\pi, -\delta]$ and $[\delta, \pi]$ as $r \uparrow 1$.

d) Show that $\int_{-\pi}^{\pi} P_r(x)\, dx = 2\pi$.

e) Let $f$ be a continuous function with period $2\pi$. Show that
$$\lim_{r\uparrow 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-y) P_r(y)\, dy = f(x).$$

f) Assume that $f$ has Fourier series $\sum_{-\infty}^{\infty} c_n e^{inx}$. Show that
$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-y) P_r(y)\, dy = \sum_{-\infty}^{\infty} c_n r^{|n|} e^{inx}$$
and that the series converges absolutely and uniformly. (*Hint:* Show that the function on the left is differentiable at $x$.)

g) Show that
$$\lim_{r\uparrow 1} \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{inx} = f(x).$$

## 10.7. Pointwise divergence of Fourier series

In this section, we shall explain why it is so hard to prove pointwise convergence of Fourier series by showing that in one sense the "normal behavior" of a Fourier series (even for a periodic, continuous function) is to diverge! The main tool will be Theorem 5.6.7 from the section on Baire's Category Theory. If you haven't read that section, you should skip this one (unless, of course, you want to go back and read it now).

As our normed space, we shall be using $C_P$ with the supremum norm $\|\cdot\|_\infty$. If

$$s_n(f)(x) = \sum_{k=-n}^{n} \langle f, e_k\rangle e_k(x)$$

denotes the partial sums of the Fourier series of $f$, we know from Section 10.3 that

$$s_n(f)(x) = \frac{1}{2\pi}\int f(x-u)D_n(u)\,du.$$

If we fix an $x \in [-\pi, \pi]$, we can think of $f \mapsto s_n(f)(x)$ as a linear functional from $C_P$ to $\mathbb{C}$. Let us denote this functional by $A_n$; hence

$$A_n(f) = \frac{1}{2\pi}\int f(x-u)D_n(u)\,du).$$

Note that $A_n$ is bounded since

$$|A_n(f)| = \frac{1}{2\pi}|\int f(x-u)D_n(u)\,du| \leq \frac{1}{2\pi}\int |f(x-u)||D_n(u)|\,du \leq K_n\|f\|_\infty,$$

where

$$K_n = \frac{1}{2\pi}\int |D_n(u)|\,du.$$

We need to know that the operator norms $\|A_n\|$ increase to infinity, and an easy way to show this, is to prove that $\|A_n\| = K_n$ (we know from Lemma 10.3.2 that $K_n \to \infty$).

**Lemma 10.7.1.** $\|A_n\| = K_n = \frac{1}{2\pi}\int |D_n(u)|\,du$.

**Proof.** From the calculations above, we know that $\|A_n\| \leq K_n$. To prove the opposite inequality, define a $2\pi$-periodic function $g$ by

$$g(x-u) = \begin{cases} 1 & \text{if } D_n(u) \geq 0 \\ \\ -1 & \text{if } D_n(u) < 0 \end{cases}$$

and note that

$$\frac{1}{2\pi}\int g(x-u)D_n(u)\,du = \frac{1}{2\pi}\int |D_n(u)|\,du = K_n.$$

Obviously, $g$ is not in $C_P$, but since $D_n$ has only finitely many zeroes, it is clearly possible to find a sequence $\{g_k\}$ of functions in $C_P$ with norm 1 that converges pointwise to $g$ in such a way that

$$|A_n(g_k)| = \frac{1}{2\pi}\int g_k(x-u)D_n(u)\,du$$

$$\to \frac{1}{2\pi}\int g(x-u)D_n(u)\,du = K_n = K_n\|g_k\|_\infty.$$

This implies that $\|A_n\| \geq K_n$ and combining our observations, we get $\|A_n\| = K_n$. $\square$

We are now ready to prove the main result.

**Theorem 10.7.2.** *Assume that $x \in [-\pi, \pi]$. The set*

$$\{f \in C_P \ : \ \text{the Fourier series of } f \text{ diverges at } x\}$$

*is comeager in $C_P$.*

**Proof.** According to the lemma, the sequence $\{A_n\}$ is not uniformly bounded (since $\|A_n\| \to \infty$), and by Theorem 5.6.7 the set of $f$'s for which $A_n(f)$ diverges, is comeager in $C_P$. As $A_n(f) = S_n(f)(x)$ is the $n$-th partial sum of the Fourier-series at $x$, the theorem follows. $\square$

As we usually think of comeager sets as "big sets", the theorem can be interpreted as saying that the normal behavior of a Fourier series is to diverge! I should add, however, that there are other results pointing in the opposite direction, e.g., a famous theorem by the Swedish mathematician Lennart Carleson (1928-) saying that the Fourier series of a square integrable function converges to the function "almost everywhere" (in a technical sense), hence indicating that the normal behavior of a Fourier series is to converge! There is no contradiction between these two statements as we are using two quite different measures of what is "normal", but they definitely show what a tricky question pointwise convergence of Fourier series is.

### Exercises for Section 10.7

1. Show that the sequence $\{g_n\}$ in the proof of Lemma 10.7.1 really exists.
2. Let $F_n$ be the Fejér kernel. Show that for each $x \in [-\pi, \pi]$,

$$B_n(f)(x) = \frac{1}{2\pi} \int f(x-u) F_n(u)\, du$$

   defines a bounded, linear operator $B_n \colon C_P \to \mathbb{C}$. Show that the sequence of norms $\{\|B_n\|\}$ is bounded.
3. a) Show that the intersection of a countable family of comeager sets is comeager.
   b) Let $T$ be a countable subset of $[-\pi, \pi]$. Show that the set

$$\{f \in C_P \ : \ \text{the Fourier series of } f \text{ diverges at all } x \in T\}$$

   is comeager.

## 10.8. Termwise operations

In Section 4.3 we saw that power series can be integrated and differentiated term by term, and we now want to take a quick look at the corresponding questions for Fourier series. Let us begin by integration which is by far the easiest operation to deal with.

The first thing we should observe, is that when we integrate a Fourier series $\sum_{-\infty}^{\infty} \alpha_n e^{inx}$ term by term, we do *not* get a new Fourier series since the constant term $\alpha_0$ integrates to $\alpha_0 x$, which is not a term in a Fourier series when $\alpha_0 \neq 0$. However, we may, of course, still integrate term by term to get the series

$$\alpha_0 x + \sum_{n \in \mathbb{Z}, n \neq 0} \left(-\frac{i\alpha_n}{n}\right) e^{inx}.$$

The question is if this series converges to the integral of $f$.

**Proposition 10.8.1.** *Let $f \in D$, and define $g(x) = \int_0^x f(t)\,dt$. If $s_n$ is the partial sums of the Fourier series of $f$, then the functions $t_n(x) = \int_0^x s_n(t)\,dt$ converge uniformly to $g$ on $[-\pi, \pi]$. Hence*

$$g(x) = \int_0^x f(t)\,dt = \alpha_0 x + \sum_{n\in\mathbb{Z}, n\neq 0} -\frac{i\alpha_n}{n}\left(e^{inx} - 1\right)$$

*where the convergence of the series is uniform.*

**Proof.** By Cauchy-Schwarz's inequality we have

$$|g(x) - t_n(x)| = |\int_0^x (f(t) - s_n(t))\,dt| \le \int_{-\pi}^{\pi} |f(t) - s_n(t)|\,dt \le$$
$$\le 2\pi\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(s) - s_n(s)|\cdot 1\,ds\right) = 2\pi\langle |f - s_n|, 1\rangle \le$$
$$\le 2\pi\|f - s_n\|_2\|1\|_2 = 2\pi\|f - s_n\|_2.$$

By Theorem 10.2.6, we see that $\|f - s_n\|_2 \to 0$, and hence $t_n$ converges uniformly to $g(x)$. □

If we move the term $\alpha_0 x$ to the other side in the formula above, we get

$$g(x) - \alpha_0 x = \sum_{n\in\mathbb{Z}, n\neq 0} \frac{i\alpha_n}{n} - \sum_{n\in\mathbb{Z}, n\neq 0} \frac{i\alpha_n}{n} e^{inx},$$

where the series on the right is the Fourier series of $g(x) - \alpha_0 x$ (the first sum is just the constant term of the series).

As always, termwise differentiation is a much trickier subject. In Example 1 of Section 10.1, we showed that the Fourier series of $x$ is

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\sin(nx),$$

and by what we now know, it is clear that the series converges pointwise to $x$ on $(-\pi, \pi)$. However, if we differentiate term by term, we get the hopelessly divergent series

$$\sum_{n=1}^{\infty} 2(-1)^{n+1}\cos(nx).$$

Fortunately, there is more hope when $f \in C_P$, i.e. when $f$ is continuous and $f(-\pi) = f(\pi)$:

**Proposition 10.8.2.** *Assume that $f \in C_P$ and that the derivative $f'$ is continuous on $[-\pi, \pi]$. If $\sum_{n=-\infty}^{\infty} \alpha_n e^{inx}$ is the Fourier series of $f$, then the differentiated series $\sum_{n=-\infty}^{\infty} in\alpha_n e^{inx}$ is the Fourier series of $f'$, and it converges pointwise to $f'$ at any point $x$ where $f''(x)$ exists.*

**Proof.** Let $\beta_n$ be the Fourier coefficient of $f'$. By integration by parts

$$\beta_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(t)e^{-int}\,dt = \frac{1}{2\pi}\left[f(t)e^{-int}\right]_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)(-ine^{-int})\,dt =$$
$$= 0 + in\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt = in\alpha_n,$$

which shows that $\sum_{n=-\infty}^{\infty} in\alpha_n e^{inx}$ is the Fourier series of $f'$. The convergence follows from Corollary 10.6.3. $\square$

**Final remark:** In this chapter we have developed Fourier analysis over the interval $[-\pi,\pi]$. If we want to study Fourier series over another interval $[a-r,a+r]$, all we have to do is to move and rescale the functions: The basis now consists of the functions

$$e_n(x) = e^{\frac{in\pi}{r}(x-a)},$$

the inner product is defined by

$$\langle f,g\rangle = \frac{1}{2r}\int_{a-r}^{a+r} f(x)\overline{g(x)}\,dx,$$

and the Fourier series becomes

$$\sum_{n=-\infty}^{\infty} \alpha_n e^{\frac{in\pi}{r}(x-a)}.$$

Note that when the length $r$ of the interval increases, the frequencies $\frac{in\pi}{r}$ of the basis functions $e^{\frac{in\pi}{r}(x-a)}$ get closer and closer. In the limit, one might imagine that the sum $\sum_{n=-\infty}^{\infty} \alpha_n e^{\frac{in\pi}{r}(x-a)}$ turns into an integral (think of the case $a=0$):

$$\int_{-\infty}^{\infty} \alpha(t)e^{ixt}\,dt.$$

This leads to the theory of Fourier integrals and Fourier transforms, but we shall not look into these topics here.

**Exercises for Section 10.8.**

1. Use integration by parts to check that $\sum_{n\in\mathbb{Z},n\neq 0} \frac{i\alpha_n}{n} - \sum_{n\in\mathbb{Z},n\neq 0} \frac{i\alpha_n}{n} e^{inx}$ is the Fourier series of $g(x) - \alpha_0 x$ (see the passage after the proof of Proposition 10.8.1).
2. Show that $\sum_{k=1}^{n} \cos((2k-1)x) = \frac{\sin 2nx}{2\sin x}$.
3. In this problem we shall study a feature of Fourier series known as *Gibbs's phenomenon*. Let $f\colon [-\pi,\pi] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ 1 & \text{for } x > 1 \end{cases}$$

   Figure 10.8.1 shows the partial sums $s_n(x)$ of order $n = 5, 11, 17, 23$. We see that although the approximation in general seems to get better and better, the maximal distance between $f$ and $s_n$ remains more or less constant — it seems that the partial sums have "bumps" of more or less constant height near the jump in function values. We shall take a closer look at this phenomenon. Along the way you will need the solution of problem 2.

   a) Show that the partial sums can be expressed as

$$s_{2n-1}(x) = \frac{4}{\pi}\sum_{k=1}^{n} \frac{\sin((2k-1)x)}{2k-1}$$

   (we did this calculation in Example 2 of section 10.1).

   b) Use problem 2 to find a short expression for $s'_{2n-1}(x)$.

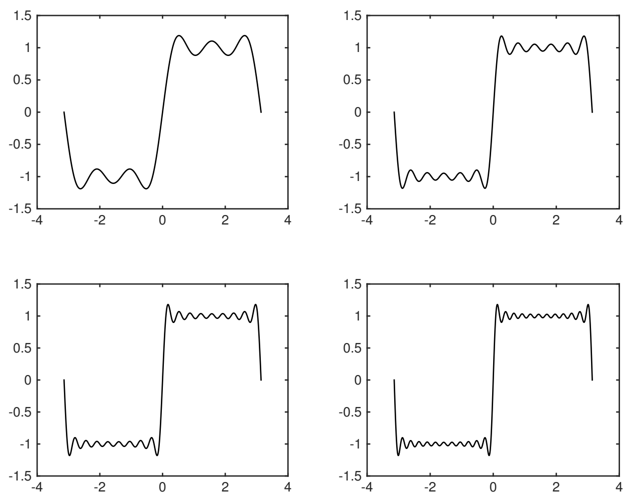


**Figure 10.8.1.** The Gibbs phenomenon

c) Show that the local minimum and maxima of $s_{2n-1}$ closest to 0 are $x_- = -\frac{\pi}{2n}$ and $x_+ = \frac{\pi}{2n}$.
d) Show that

$$s_{2n-1}(\pm\frac{\pi}{2n}) = \pm\frac{4}{\pi}\sum_{k=1}^{n}\frac{\sin\frac{(2k-1)\pi}{2n}}{2k-1}.$$

e) Show that $s_{2n-1}(\pm\frac{\pi}{2n}) \to \pm\frac{2}{\pi}\int_0^\pi \frac{\sin x}{x}\,dx$ by recognizing the sum above as a Riemann sum.
f) Use a calculator or a computer or whatever you want to show that $\frac{2}{\pi}\int_0^\pi \frac{\sin x}{x}\,dx \approx 1.18$.

These calculations show that the size of the "bumps" is 9% of the size of the jump in the function value. Gibbs showed that this number holds in general for functions in $D$.

## Notes and references to Chapter 10

There is an excellent account of the discussion of the vibrating string in Katz' book [**20**]. It influenced not only the development of Fourier analysis, but also the understanding of the function concept.

Jean Baptiste Joseph Fourier (1768-1830) published his first treatise on heat propagation in 1807 and a second one in 1822. Although Fourier himself was mainly interested in applications in physics, his theory was soon brought to bear on problems in pure mathematics, and in 1837 Johann Peter Gustav Lejeune Dirichlet (1805-1859) used it to prove that any sequence $\{an+b\}_{n\in\mathbb{N}}$ where $a, b \in \mathbb{N}$ are relatively prime, contains infinitely many primes. The Fejér kernel is named after the Hungarian mathematician Lipót Fejér (1880-1959) who proved the Cesàro convergence of Fourier series at the age of 20. Dini's Test was proved by Ulisse Dini (1845-1918) in 1880.

Körner's book [**22**] contains a wealth of material on Fourier analysis and applications. The text by Stein and Shakarchi [**37**] is more geared toward applications in other parts of mathematics – it it's the first part of a four volume series (*Princeton Lectures in Analysis*) that is highly recommended. Montgomery's book [**28**] is a little slower and more elementary, but contains lots of interesting examples and applications, and the old book by Tolstov [**41**] is still eminently readable.

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