Exercises week 18 (Wed 22.04)

Ulrich's exercise Let X, Y, Z be normed spaces, and $F: X \to Y \times Z$ a function $F(x) = (G(x), H(x))$.

Show that

F Frechet diff. $\Leftrightarrow$ G and H Frechét diff.

Moreover, show that $F'(x) = (G'(x), H'(x))$.

Proof. "$\Rightarrow$": If F is differentiable, $F'(x)$ is a linear map $F'(x): X \to Y \times Z$. So $F'(x) = (A(x), B(x))$ for some bounded linear maps $A(x): X \to Y$, $B(x): X \to Z$.

Write
$$\begin{aligned}\vartheta_F(r) &= F(a+r) - F(a) - F'(a)(r) \\ &= (G(a+r) - G(a) - A(a)(r), H(a+r) - H(a) - B(a)(r)) \\ &=: (\vartheta_G(r), \vartheta_H(r)).\end{aligned}$$

We get

$$\frac{\|\vartheta_F(r)\|_{Y\times Z}}{\|r\|_X} = \frac{\|\vartheta_G(r)\|_Y}{\|r\|_X} + \frac{\|\vartheta_H(r)\|_Z}{\|r\|_X} \qquad (*)$$

This goes to 0 as $r \to 0$, so both expressions on the right hand side goes to 0 as $r \to 0$.

Hence G and H are diff. with $G'(x) = A(x)$, $H'(x) = B(x)$

"$\Leftarrow$": Still get equation $(*)$. The left hand side tends to 0 $\Rightarrow$ $\frac{\|\vartheta_F(r)\|}{\|r\|} \to 0$. $\Rightarrow$ F diff'able.

∎

6.8.2. Note that $f(-1,2,0) = -4$. Moreover,

$$\frac{\partial f}{\partial x}(x,y,z) = y^2 e^z, \quad \frac{\partial f}{\partial y}(x,y,z) = 2xye^z, \quad \frac{\partial f}{\partial z}(x,y,z) = xy^2e^z + 1.$$

Let $h : \mathbb{R}^3 \to \mathbb{R}$ be the function $h = f + 4$.

Then, $h(-1,2,0) = 0$, and h has continuous partial derivatives in a nbh. around $(-1,2,0)$. Also

$$\frac{\partial h}{\partial z}(-1,2,0) = \frac{\partial f}{\partial z}(-1,2,0) = -3 \neq 0.$$

By the Implicit function theorem (or Cor. 6.8.3) there is a nbh. V around $(-1,2) \in \mathbb{R}^2$ and a differentiable function $g : V \to \mathbb{R}$ s.t.

$$h(x,y,g(x,y)) = 0 \quad \forall\, x,y \in V$$

$$\iff f(x,y,g(x,y)) = -4 \quad \forall\, x,y \in V.$$

Moreover

- $$\frac{\partial g}{\partial x}(-1,2) = -\frac{\frac{\partial h}{\partial x}(-1,2,\overset{0}{\overset{\|}{g(-1,2)}})}{\frac{\partial h}{\partial z}(-1,2,g(-1,2))} = -\frac{1}{-3} = \frac{1}{3}$$

- $$\frac{\partial g}{\partial y}(-1,2) = -\frac{\frac{\partial h}{\partial y}(-1,2,0)}{\frac{\partial h}{\partial z}(-1,2,0)} = -\frac{-4}{-3} = -\frac{4}{3}$$

6.8.5. a) Note that $F(1, \frac{\pi}{2}, 1) = 2$. Moreover

- $\frac{\partial F}{\partial x}(x,y,z) = \sin(xyz^2) + xyz^2\cos(xyz^2)$
- $\frac{\partial F}{\partial y}(x,y,z) = x^2z^2\cos(xyz^2)$
- $\frac{\partial F}{\partial z}(x,y,z) = 2x^2yz\cos(xyz^2) + 1$

Put $G(x,y,z) = F(x,y,z) - 2$. Then

$G(1, \pi/2, 1) = 0$, and G has the same (continuous) partial derivatives as F.

$\frac{\partial H}{\partial z}(1, \pi/2, 1) = 1 \neq 0$, so by the Implicit function theorem, $\exists$ a neighborhood V around $(1, \pi/2)$ and a differentiable function $Z : V \to \mathbb{R}$, s.t.

$$Z(1, \pi/2, 1) = 1 \quad \text{and} \quad H(x,y,Z(x,y)) = 0$$

for all $x,y \in V$.

b) By Cor. 6.8.3, Z is differentiable at $(1, \pi/2)$ with partial derivative

$$\frac{\partial Z}{\partial x}(1, \pi/2) = -\frac{\frac{\partial F}{\partial x}(1, \pi/2, 1)}{\frac{\partial F}{\partial z}(1, \pi/2, 1)} = -\frac{1}{1} = -1$$ //

Rest of the semester!

6.6.4. Let $y \in X$, and $F_y : X \to \mathbb{R}$, $F_y(v) = \langle v, y \rangle$.

F_y is a bounded linear operator, so

$$\frac{\partial F}{\partial x}(x,y)(v) \underset{\substack{\uparrow \\ \text{Def. of partial} \\ \text{derivative}}}{=} F_y'(x)(v) \underset{\substack{\uparrow \\ \text{Prop 6.1.5.}}}{=} F_y(v) = \langle v, y \rangle .$$

Now, let $x \in X$, and $G_x : X \to \mathbb{R}$, $G_x(v) = \langle x, v \rangle$.

As X is a real inner product space,

$$G_x(v) = \langle x, v \rangle = \langle v, x \rangle = F_x(v).$$

So $\frac{\partial F}{\partial y}(x,y)(v) = G_x'(x,y)(v) = \langle x, v \rangle$ by the above. (In the notation of the book $F_y = F^1_{(x,y)}$, $G_x = F^2_{(x,y)}$)

6.6.6. Let $(t, f) \in \mathbb{R} \times C([0,1], \mathbb{R})$. Then

- $\left(\frac{\partial F}{\partial t}(t,f)\right)(t) = \left(F^1_{(t,f)}\right)'(t) = \frac{d}{dt}\left(\int_0^t f(s)\,ds\right)$

$$\underset{\boxed{\text{Fundamental thm. of calculus.}}}{=} f(t)$$

- Note that $F^1_{(t,f)}(f) = \int_0^t f(s)\,ds = i_t(f)$.

As this is a linear map

$$\frac{\partial F}{\partial f}(t,f) = i_t'(f) = i_t .$$

6.8.12. a) By assumption there is a point (x_0, y_0) such that
- $\frac{\partial F}{\partial y}(x_0, y_0)$ is a bijection
- $F(x_0, y_0) = 0$

Also F is differentiable. $\rightsquigarrow$ Implicit function theorem.

b) $$h'(x_0) = \underbrace{f'(x_0, y_0)}_{=0} \circ (I_X, G'(x_0))$$

c) $$f'(x_0, y_0)(v_1, v_2) = \frac{\partial f}{\partial x}(x_0, y_0)(v_1) + \frac{\partial f}{\partial y}(x_0, y_0)(v_2)$$

$$\rightsquigarrow h'(x_0) = \frac{\partial f}{\partial x}(x_0, y_0) + \frac{\partial f}{\partial y}(x_0, y_0)\, G'(x_0)$$

by the chain rule.

d). $\frac{\partial F}{\partial y}(x_0, y_0) : Y \to Z$ is a linear map

$\Rightarrow \left[\frac{\partial F}{\partial y}(x_0, y_0)\right]^{-1} : Z \to Y$ is a linear map

- $\frac{\partial f}{\partial y}(x_0, y_0) : Y \to \mathbb{R}$ is a linear map

$$\Rightarrow \lambda = \frac{\partial f}{\partial y}(x_0, y_0) \circ \left[\frac{\partial F}{\partial y}(x_0, y_0)\right]^{-1} : Z \to \mathbb{R}$$

is a linear map. Moreover

$$\frac{\partial f}{\partial x}(x_0, y_0) = -\frac{\partial f}{\partial y}(x_0, y_0)\, G'(x_0)$$

$$\overset{a)}{=} -\frac{\partial f}{\partial y}\left(-\left[\frac{\partial F}{\partial y}(x_0, y_0)\right]^{-1} \circ \frac{\partial F}{\partial x}(x_0, y_0)\right)$$

$$\overset{\text{above}}{=} \lambda \circ \frac{\partial F}{\partial x}(x_0, y_0)$$

6.8.12. e) Multiplying the equation in d) by $\frac{\partial F}{\partial y}(x_0,y_0)$ on the right, we get

$$\frac{\partial f}{\partial y}(x_0,y_0) = \lambda \circ \frac{\partial F}{\partial y}(x_0,y_0).$$

$$\rightsquigarrow f'(x_0,y_0)(v_1,v_2) = \frac{\partial f}{\partial x}(x_0,y_0)(v_1) + \frac{\partial f}{\partial y}(x_0,y_0)(v_2)$$

$$\boxed{\text{By d) and above}} \overset{\downarrow}{=} \lambda \circ \frac{\partial F}{\partial x}(x_0,y_0)(v_1) + \lambda \circ \frac{\partial F}{\partial y}(x_0,y_0)(v_2)$$

$$= [\lambda \circ F'(x_0,y_0)](v_1,v_2).$$

f) Google "Lagrange multiplier" or find a book on the topic (for instance the one used in MAT1110).

When $Y = Z = \mathbb{R}$, the last equation in e) becomes

$$\nabla_{x,y,z}\left(f(x,y) - \lambda F(x,y)\right) = 0$$

and for $Y = Z = \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$ and $F = (F_1, F_2, \ldots, F_n) : X \times \mathbb{R}^n \to \mathbb{R}^n$, so the equation becomes

$$\nabla_{\vec{x},\lambda}\left(f(\vec{x}) - \sum_{j=1}^{n} \lambda_j F_1(\vec{x})\right).$$

10.1.5. For $n \in \mathbb{Z}$, $e_n = e^{inx}$.

$$\int_{-\pi}^{\pi} e^x e^{-inx}\,dx = \int_{-\pi}^{\pi} e^{(1-in)x}\,dx$$

$$= \frac{1}{1-in} e^{(1-in)x}\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{1-in}\left(e^{(1-in)\pi} - e^{-(1-in)\pi}\right)$$

$$\boxed{e^{in\pi} = \begin{cases} -1 & : n \text{ odd} \\ 1 & : n \text{ even} \end{cases}}$$

$$\downarrow = \frac{(-1)^n}{1-in}\left(e^{\pi} - e^{-\pi}\right)$$

The Fourier series of e^x is $\sum_{n=-\infty}^{\infty} \alpha_n e_n$, where

$\alpha_n = \frac{(-1)^n}{2\pi(1-in)}\left(e^{\pi} - e^{\pi}\right)$. So

$$\sum_{n=-\infty}^{\infty} \alpha_n e_n(x) = \frac{e^{\pi}-e^{-\pi}}{2\pi}\left(1 + \sum_{n=-\infty}^{-1} \frac{(-1)^n}{(1-in)} e^{inx} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(1-in)} e^{inx}\right)$$

$$= \frac{e^{\pi}-e^{-\pi}}{2\pi}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{1+in} e^{-inx} + \sum_{n=1}^{\infty} \frac{(-1)^n}{1-in} e^{inx}\right)$$

$$= \frac{e^{\pi}-e^{-\pi}}{2\pi}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n\left(in\left(e^{inx} - e^{-inx}\right) + e^{inx} + e^{-inx}\right)}{1+n^2}\right)$$

$$= \frac{e^{\pi}-e^{-\pi}}{2\pi} + \frac{e^{\pi}-e^{-\pi}}{2\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}\left(2\cos(nx) - 2n\sin(nx)\right).$$

10.1.7. $\cdot\ \alpha_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin\left(\frac{x}{2}\right) dx = 0$ 

$$\cdot\ 2\pi\alpha_n = \int_{-\pi}^{\pi} \sin\left(\frac{x}{2}\right) e^{-inx}\, dx \overset{(*)}{=} \frac{8in(-1)^n}{4n^2-1}.$$

$$\sum_{n=-\infty}^{\infty} \alpha_n e_n(x) = \frac{8}{2\pi}\sum_{n=-\infty}^{\infty}\left(\frac{in(-1)^n}{4n^2-1}\right)e^{-inx}$$

$$= \frac{4}{\pi}\left(\sum_{n=1}^{\infty}\frac{in}{4n^2-1}\left[(-1)^{n+1}e^{inx} + (-1)^n e^{-inx}\right]\right)$$

$$= \frac{2}{\pi}\left(\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\, 2i\left(e^{inx} - e^{-inx}\right)\right)$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n\sin(nx)}{4n^2-1}$$

(So $a_n = 0$, $b_n = \frac{2}{\pi}\frac{n}{4n^2-1}$)

$$(*)\ I = \int_{-\pi}^{\pi}\sin\left(\frac{x}{2}\right)e^{-inx}dx = -2\cos\left(\frac{x}{2}\right)e^{-inx}\Big|_{-\pi}^{\pi} - 2in\int_{-\pi}^{\pi}\cos\left(\frac{x}{2}\right)e^{-inx}dx$$

$$= -2in\left[2\sin\left(\frac{x}{2}\right)e^{-inx}\Big|_{-\pi}^{\pi} + 2in\int_{-\pi}^{\pi}\sin\left(\frac{x}{2}\right)e^{-inx}dx\right]$$

$$= -4in\left(e^{in\pi} + e^{-in\pi}\right) + 4n^2 I$$

$$\Rightarrow I - 4n^2 I = -4in\cdot 2\cos(n\pi)$$

$$\Rightarrow I = -\frac{8in\cos(n\pi)}{1-4n^2} = \frac{8in(-1)^n}{4n^2-1}$$

(Note: You can possibly save some time by only computing b_n)

10.1.10. a) Let $g : \mathbb{R} \to \mathbb{R}$.

$$\int_{-a}^{a} g(x)\,dx = \int_{-a}^{0} g(x)\,dx + \int_{0}^{a} g(x)\,dx$$

$$= \int_{0}^{a} g(-x)\,dx + \int_{0}^{a} g(x)$$

$$= \begin{cases} 2\int_0^a g(x)\,dx & \text{if } g \text{ is even} \\ 0 & \text{if } g \text{ is odd} \end{cases}$$

Note that

- $f(x)\cos(nx)$ is $\begin{cases} \text{even if } f \text{ is even} \\ \text{odd if } f \text{ is odd} \end{cases}$

- $f(x)\sin(nx)$ is $\begin{cases} \text{odd if } f \text{ is even} \\ \text{even if } f \text{ is odd.} \end{cases}$

Moreover

$$\alpha_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx$$

$$= \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx}_{\frac{1}{2}a_n} - \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx}_{\frac{1}{2}b_n}$$

$\rightsquigarrow$ The claim.

b) As $|x|$ is even, it suffices to compute

$$\underset{n\neq 0}{a_n} = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos(nx)\, dx = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\, dx$$

$$= \frac{2}{\pi}\left[x \cdot \frac{-1}{n}\sin(nx)\Big|_{-\pi}^{\pi} + \frac{1}{n}\int_0^{\pi} \sin(nx)\, dx \right]$$

$$= \frac{2}{\pi}\left(0 - \frac{1}{n^2}\left[\cos(nx)\right]_0^{\pi} \right)$$

$$= \frac{2}{\pi n^2}\left(1 - \cos(n\pi)\right)$$

$$1 - \cos(n\pi) = \begin{cases} 2 & : n \text{ odd} \\ 0 & : n \text{ even} \end{cases}$$

$$\Rightarrow \quad a_n = \frac{4}{\pi n^2} \quad n \text{ odd}, \quad a_n = 0, \quad n \text{ even}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\, dx = \frac{2}{\pi}\frac{1}{2}\pi^2 = \pi$$

$$\rightsquigarrow \quad \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) = \frac{\pi}{2} + \frac{4}{\pi}\left(\cos x + \frac{\cos(3x)}{3^2} + \dots\right.$$

c) We must compute a_n again.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin(x)|\, dx = \frac{2}{\pi}\int_0^{\pi} \sin(x)\, dx$$

$$= \frac{2}{\pi}\left(-\cos(\pi) + \cos(0)\right) = 4/\pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin x| \cos(nx)\, dx = \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(nx)\, dx$$

$\rightsquigarrow$

c forts)

$$\overset{\text{Hint}}{=} \frac{1}{\pi}\int_0^\pi \sin[(n+1)x]\,dx - \frac{1}{\pi}\int_0^\pi \sin[(n-1)x]\,dx$$

$$= \frac{1}{\pi}\left[-\frac{1}{n+1}\left(\cos[(n+1)\pi]-1\right) + \frac{1}{n-1}\left(\cos[(n-1)\pi]-1\right)\right]$$

n even $\rightsquigarrow$ $a_n = \frac{1}{\pi}\left(\frac{2}{n+1}-\frac{2}{n-1}\right)$

$$= \frac{1}{\pi}\left(\frac{2(n-1)-2(n+1)}{n^2-1}\right)$$

$$= \frac{1}{\pi}\left(\frac{-4}{n^2-1}\right)$$

n odd $\rightsquigarrow$ $a_n = 0$ because $\cos((n\pm1)\pi)-1=0$.

(The case $n=1$ should be considered by itself.)

$$\Rightarrow \frac{a_0}{2} + \sum_{n=1}^\infty a_n\cos(nx) = \frac{2}{\pi} + \frac{(-4)}{\pi}\sum_{n=1}^\infty \frac{\cos(2nx)}{n^2-1}.$$

(To prove the hint, write $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$

and $\cos(nx) = \frac{e^{inx}+e^{-inx}}{2}$, and rewrite

$2\sin x\cos(nx)$ until you get

$$\frac{e^{i(n+1)x}-e^{-i(n+1)x}}{\quad} - \frac{e^{i(n-1)x}-e^{-i(n-1)x}}{\quad})$$

d) $f_e(-x) = \dfrac{f(-x)+f(x)}{2} = f_e(x)$, so f_e is even. Write a_n^e for the Fourier coeff's of f_e.

$$\pi a_n^e = \int_{-\pi}^{\pi} f_e(x)\cos(nx)\,dx$$

Def of f_e, and f_e is even

$$\stackrel{\downarrow}{=} \int_0^{\pi}\big(f(x)+f(-x)\big)\cos(nx)\,dx$$

$$= \int_0^{\pi} f(x)\cos(nx)\,dx + \underbrace{\int_0^{\pi} f(-x)\cos(nx)\,dx}_{=I}$$

$$I = -\int_0^{-\pi} f(x)\cos(-nx)\,dx = \int_{-\pi}^{0} f(x)\cos(nx)\,dx$$

$$\Rightarrow \pi a_n^e = \int_0^{\pi} f(x)\cos(nx)\,dx + \int_{-\pi}^{0} f(x)\cos(nx)\,dx$$

$$= \int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \pi a_n.$$

$b_n^e = b_n$ is shown in a similar way.

10.2.1. Let $f_n \in C_p$ be functions converging to $f \in C([-\pi,\pi],\mathbb{C})$. We must show that $f(\pi) = f(-\pi)$.

But $|f(\pi) - f(-\pi)| = |f(\pi) + f_n(\pi) - f_n(-\pi) - f(-\pi)|$

$$\leq |f(\pi) - f_n(\pi)| + |f_n(-\pi) - f(\pi)|$$

$$\longrightarrow 0 \quad \text{as } n \to \infty$$

by assumption, because

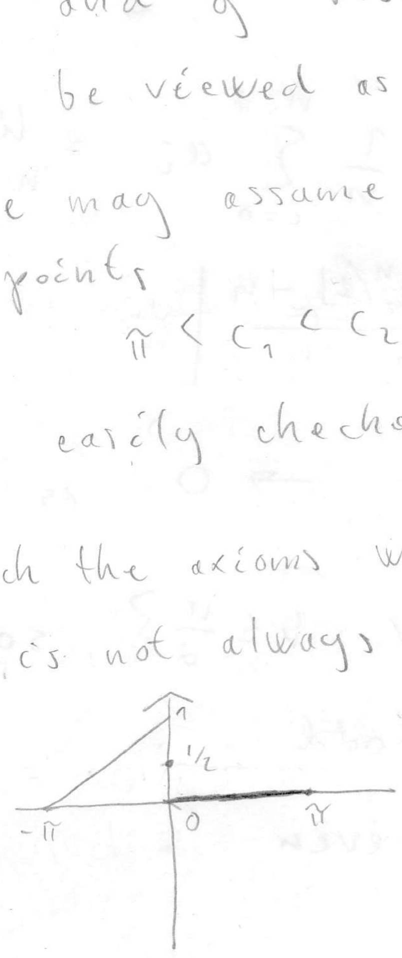
uniform convergence $\Rightarrow$ pointwise convergence.

10.2.2. a) Let $-\pi < a_1 < a_2 < \dots < a_n < \pi$ and $-\pi < b_1 < b_2 < \dots < b_k < \pi$ be the "jump points" for f and g respectively. Then both f and g can be viewed as having all these "jump points". So, we may assume f and g have the same jump points

$$\pi < c_1 < c_2 < \dots < c_m < \pi, \text{ then}$$

it is easily checked that $f+g \in D$.

b) Check the axioms when $f \in D$, $g \in C_p$. To see that fg is not always in D for $f, g \in D$ consider

$f =$ 

Then $f^2 = ff$ satisfies

- $f^2(0^+) = 0$
- $f^2(0^-) = 1$

So $\frac{1}{2}(f^2(0^+) + f^2(0^-)) = \frac{1}{2} \neq \frac{1}{4} = f^2(0)$.

10.2.2 c) $0 \in D$, we can add functions in D (by a)), and it is easy to see that we can multiply elements of D by numbers in $\mathbb{C}$.

d) $f \in D$. $\sup_{x \in [-\pi, \pi]} |f(x)| = \max_i \sup_{x \in (a_i, a_{i+1})} |f(x)| < \infty$,

as f is cont. on (a_i, a_{i+1}) and $|f(a_i^+)|, |f(a_i^-)| < \infty$.

e) By d), $f \in D$ is piecewise cont. and bounded, so f is integrable. (The proof of this can probably be found in most Calculus books.)

g) Check the axioms. (Note that $f\bar{g}$ might not be in D, but it's still bounded and piecewise cont.)

10.4.1. $C\text{-}\lim_{n \to \infty} a_n := \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} a_i = \lim_{n \to \infty} \frac{1}{n} \lfloor n/2 \rfloor$

$$\left| \frac{1}{n}(\lfloor n/2 \rfloor) - \frac{1}{2} \right| = \left| \frac{2\lfloor n/2 \rfloor - n}{2n} \right|$$

$$\le \frac{1}{2n} \to 0 \quad \text{as } n \to \infty.$$

Here $\lfloor n/2 \rfloor = \max\{k \in \mathbb{N} : k \le \frac{n}{2}\}$, so

$$2\lfloor n/2 \rfloor = \begin{cases} n-1 & : n \text{ odd} \\ n & : n \text{ even} \end{cases}.$$

10.4.2. Write $S_n^a = \frac{1}{n}\sum_{i=0}^{n-1} a_i$, $S_n^b = \frac{1}{n}\sum_{i=0}^{n-1} b_i$,

$a = \lim_n S_n^a = C\text{-}\lim_n a_n$, $b = \lim_n S_n^b = C\text{-}\lim_n b_n$.

Then $\left| a + b - \frac{1}{n}\sum_{i=0}^{n-1}(a_i + b_i)\right|$

$$= \left| a + b - \frac{1}{n}\sum_{i=0}^{n-1} a_i - \frac{1}{n}\sum_{i=0}^{n-1} b_i \right|$$

$$= \left| a + b - S_n^a - S_n^b \right|$$

$$\leq |a - S_n^a| + |b - S_n^b| \to 0$$

by assumption.

10.4.3. $\lim_{u\to 0} \frac{\sin^2\left(\frac{nu}{2}\right)}{n \sin^2\left(\frac{u}{2}\right)} = ?$

Note that

$$\lim_{u\to 0} \frac{\sin\left(\frac{nu}{2}\right)}{\sin\left(\frac{u}{2}\right)} \overset{\left[\frac{0}{0}\right]}{=} \lim_{u\to 0} \frac{n\cos\left(\frac{nu}{2}\right)}{\cos\left(\frac{u}{2}\right)} = n.$$

Hence

$$\frac{\sin^2\left(\frac{nu}{2}\right)}{n\sin^2\left(\frac{u}{2}\right)} = \frac{1}{n}\left(\frac{\sin\left(\frac{nu}{2}\right)}{\sin\left(\frac{u}{2}\right)}\right)^2 \underset{u}{\to} \frac{1}{n} n^2 = n.$$

10.4.6.

a) The Dirichlet kernels fails (ii) by lem. 10.3.2.

The Fejér kernels:

(i) Prop 10.4.4.

(ii) $$\int_{-\pi}^{\pi} |F_n(x)|\, dx = \int_{-\pi}^{\pi} F_n(x)\, dx = 1$$

by Prop 10.4.4.

(iii) $$\int_{-\pi}^{-\delta} F_n(x)\, dx \overset{F_n \text{ is even}}{=} \int_{\delta}^{\pi} F_n(x)\, dx$$

$$\overset{10.4.4}{\leq} \frac{\pi^2}{n^2} \int_{\delta}^{\pi} \frac{1}{x^2}\, dx$$

$$= \frac{\pi^2}{n^2} \left(\frac{1}{\delta} - \frac{1}{\pi} \right) \underset{n}{\to} 0 .$$

b) $$S_n(x) - f(x) \overset{a(i)}{=} S_n(x) - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) K_n(x)\, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{[f(x-u) - f(x)] K_n(u)}_{H(x,u)}\, du$$

$$= \frac{1}{2\pi} \left[\underbrace{\int_{-\pi}^{-\delta} H(x,u)\, du}_{I_1(x)} + \underbrace{\int_{-\delta}^{\delta} H(x,u)\, du}_{I_2(x)} + \underbrace{\int_{\delta}^{\pi} H(x,u)\, du}_{I_3(x)} \right]$$

where $0 < \delta < \pi$.

10.4.6. forts.

- $$\|I_1\|_\infty = \left\|\frac{1}{2\pi}\int_{-\pi}^{-\delta} (f(x-u) - f(x)) K_n(u)\,du\right\|_\infty$$
$$\leq \frac{\|f\|_\infty}{\pi}\int_{-\pi}^{-\delta} |K_n(u)|\,du \to 0$$

- $$\|I_3\|_\infty \leq \frac{\|f\|_\infty}{\pi}\int_{\delta}^{\pi} |K_n(u)|\,du \to 0$$

- For any $\varepsilon > 0$, we may take $\delta > 0$ s.t. $|f(x-t) - f(x)| < \varepsilon$ when $-\delta < t < \delta$.

Then
$$\|I_2\|_\infty = \left|\frac{1}{2\pi}\int_{-\delta}^{\delta} (f(x-u) - f(x)) K_n(u)\,du\right|$$
$$\leq \frac{1}{2\pi}\varepsilon\int_{-\delta}^{\delta} |K_n(u)|\,du$$
$$\leq \frac{M}{2\pi}\varepsilon.$$

Something like this. take a look at the proof of 10.4.5.