2.1.1 $x_n \xrightarrow{n} a$ $\iff$ for all $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $|x_n - a| < \varepsilon/|M|$ when $n \geq N$.

Hence for all $n \geq N$:

$$|Mx_n - Ma| = |M||x_n - a| < |M| \varepsilon/|M| = \varepsilon.$$

So $Mx_n \xrightarrow{n} Ma$

2.1.2. $\boxed{\text{Note: If } x \in \mathbb{R},\ a \in [0,\infty), \text{ then } |x| \leq a \iff -a \leq x \leq a.}$

Assume $(x_n)_n$ and $(z_n)_n$ converge to a. Fix $\varepsilon > 0$.
By assumption we can find $N, M \in \mathbb{N}$ such that

- $|x_n - a| < \varepsilon$ for $n \geq N$
- $|z_n - a| < \varepsilon$ for $n \geq M$.

Let $K = \max\{N, M\}$. Then

$$-\varepsilon < x_n - a \leq y_n - a \leq z_n - a < \varepsilon$$

for $n \geq K$. So $|y_n - a| < \varepsilon$ for $n \geq K$. ∎

2.1.4. $\varepsilon > 0$. Can find $\delta_1, \delta_2 > 0$ s.t.

- $|x - a| < \delta_1 \implies |f(x) - f(a)| < \varepsilon/2$
- $|x - a| < \delta_2 \implies |g(x) - g(a)| < \varepsilon/2$

Let $\delta = \max\{\delta_1, \delta_2\}$. Then

$$|x - a| < \delta \implies |(f+g)(x) - (f+g)(a)| \leq |f(x) - f(a)| + |g(x) - g(a)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

2.1.7. • $\|a\| = \|a - b + b\| \leq \|a - b\| + \|b\|$

$$\Rightarrow \|a\| - \|b\| \leq \|a - b\|$$

• $\|b\| = \|b - a + a\| \leq \|b - a\| + \|a\|$

$$\Rightarrow \|b\| - \|a\| \leq \|b - a\| = \|a - b\|,$$

So $$-\|a - b\| \leq \|a\| - \|b\| \leq \|a - b\|$$

$$\Leftrightarrow \big|\|a\| - \|b\|\big| \leq \|a - b\|.$$ ▨

2.1.1. Obvious. (If $A \subset \mathbb{R}$, $\sup A$ is the smallest number $\geq$ all numbers in A)

2.1.2. Look at the proof of the prop.

2.1.3. ——— " ———

2.1.4. For each $n \in \mathbb{N}$, let $M_n = \sup \underbrace{\{a_k : k \geq n\}}_{A_n}$, $m_n = \inf \{a_k : k \geq n\}$, then, by def.

• $\limsup_{n \to \infty} a_n = \lim_{n \to \infty} M_n$

• $\liminf_{n \to \infty} a_n = \lim_{n \to \infty} m_n$

Clearly $\limsup_{n \to \infty} (-1)^n = 1$, $\liminf_{n \to \infty} (-1)^n = -1$, because $A_n = \{1, -1\}$ for all n. $(A_n = \{a_k : k \geq n\})$

2.1.5. Hint: $A_n = \{1, 0\}$ in this case.

2.2.9. Let $A_\varepsilon = \{f(x) : x \in (a-\varepsilon, a+\varepsilon), x \neq a\}$.

a) If $\varepsilon' < \varepsilon$, $A_{\varepsilon'} \subset A_\varepsilon$. So the number $\sup A_{\varepsilon'}$ does not have to be larger than the elements in $A_\varepsilon \setminus A_{\varepsilon'}$. Hence $\sup A_{\varepsilon'} \leq \sup A_\varepsilon$

Similar for inf.

b) Compare with Theorem 2.2.2.

c) Compare with Prop 2.2.3. Look also at exercise 2.1.2.

2.2.10. Did this in lecture. (It is also a very well known and important result, so you can easily find a proof somewhere I guess.)

2.3.1. Hint: Consider $\sqrt{2}$

2.3.2. —— " ——

2.3.4. Consider

$$x_1 = 1, \quad x_2 = \frac{14}{10}, \quad x_3 = \frac{141}{100}, \quad x_4 = \frac{1414}{1000}, \quad x_5 = \dots$$

$$x_n \to \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}.$$

2.3.9. Hint: Mean value theorem. (2.3.7.)

3.1.1. Check the axioms of a metric.

3.1.2. ——— " ———

3.1.6.

- Positivity follows from (i)
- Symmetry follows since $x-y=y-x$ in a vector space.
- The triangle inequality is (iii).

3.1.7. Induction:

$n=3$: $d(x_1,x_3) \leq d(x_1,x_2)+d(x_2,x_3)$ by the triangle inequality

Fix $n\in\mathbb{N}$. Suppose $d(x_1,x_n) \leq d(x_1,x_2)+\dots+d(x_{n-1},x_n)$.

If $x_{n+1}\in X$, we have

$$d(x_1,x_{n+1}) \leq d(x_1,x_n)+d(x_n,x_{n+1}) \leq d(x_1,x_2)+d(x_2,x_3)+\dots+d(x_n,x_{n+1})$$

Induction hypothesis,

☐

3.2.1. If $x_n=a$ $\forall n$ larger than some $N\in\mathbb{N}$, then $d(x_n,a)=0$ for all $n\geq N$, (and $0<\varepsilon$).

If $x_n\to a$, then we can find $N\in\mathbb{N}$ s.t. $d(x_n,a)<1$ for all $n\geq N$. But then $d(x_n,a)=0$, so $x_n=a$ for all $n\geq N$ by the first axiom of a metric space (positivity).

☐

3.2.2. $\varepsilon > 0$. As g is continuous at $f(a)$ we can find $\delta > 0$ s.t. $|g(y) - g(f(a))| < \varepsilon$ for all $y \in Y$ s.t. $|y - f(a)| < \delta$. As f is cont. we can find δ' s.t. $|f(x) - f(a)| < \delta$ for all x s.t. $|x - a| < \delta'$. Hence δ' is such that

$$|x - a| < \delta' \Rightarrow |g(f(x)) - g(f(a))| < \varepsilon.$$

Hence $g \circ f$ is continuous. $\blacksquare$

$(|x - a| < \delta' \Rightarrow |f(x) - f(a)| < \delta \Rightarrow |g(f(x)) - g(f(a))| < \varepsilon)$

3.2.5. $\varepsilon > 0$. We check that f is cont. at $y \in X$. Set $\delta = \varepsilon$. Then if $d(x,y) < \delta = \varepsilon$

$$|d(x,a) - d(y,a)| \leq d(x,y) < \varepsilon$$

by the inverse triangle inequality.

3.2.6. Hint: Try $\delta = \varepsilon / K$.

3.2.8. a) Hint: Use Prop 3.2.5 and exercise 3.2.5.

b) Note: $(d(x_n, y_n))_n$ is a sequence in $\mathbb{R}$. Use that

$$\begin{aligned}|d(x_n,y_n) - d(x,y)| &= |d(x_n,y_n) - d(x_n,y) + d(x_n,y) - d(x,y)| \\ &\leq |d(x_n,y_n) - d(x_n,y)| + |d(x_n,y) - d(x,y)| \\ &\leq d(y_n,y) + |d(x_n,y) - d(x,y)|.\end{aligned}$$

Inverse triangle ineq.

Now, use that $x_n \to x$, $y_n \to y$ and a).