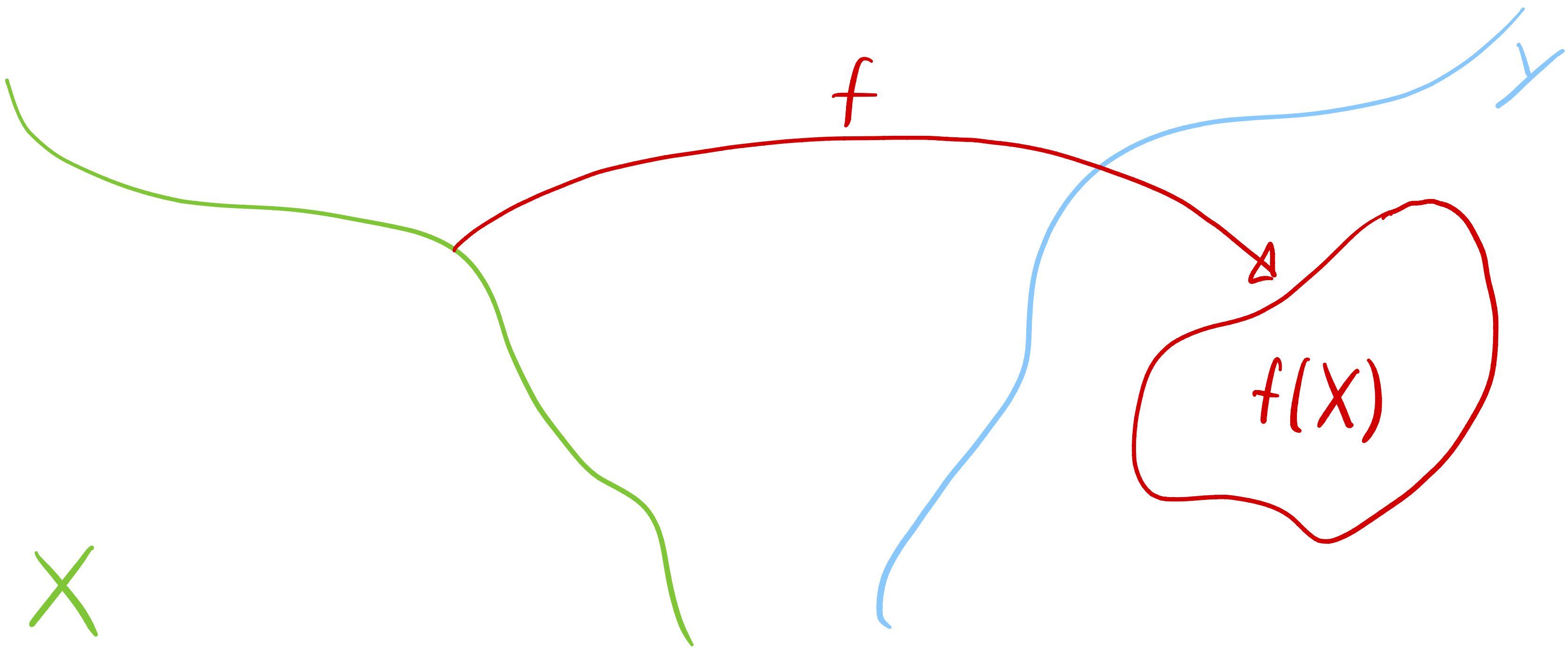
THE SPACE OF BOUNDED FUNCTIONS

Let (X, d_X) and (Y, d_Y) be metric spaces.

A function $f: X \to Y$ is <u>bounded</u> if $f(X)$ is bounded.



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A function $f : X \to Y$ is <u>bounded</u> if $f(X)$ is bounded.

Let $B(X, Y) = \{\text{all bounded } f : X \to Y\}$

Recall: The supremum metric is $\rho(f,g) = \sup_{x \in X} d_Y(f(x), g(x))$.

Recall: $\rho(f_n, f) \xrightarrow[n\to\infty]{} 0 \iff f_n \xrightarrow[n\to\infty]{} f$ uniformly.

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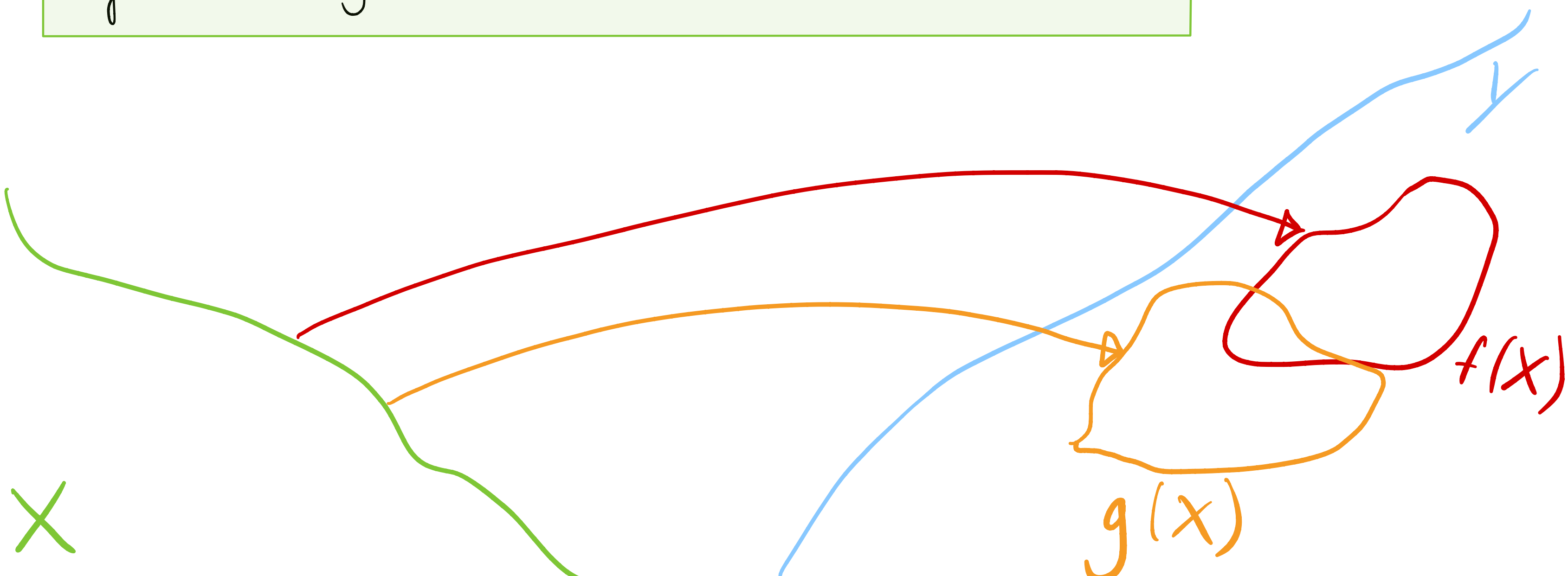
Exercise

If f and g are bounded then $\rho(f,g) < \infty$.



Proposition: ρ is a metric on $B(X,Y)$.

Proof: • First of all, $\rho(f,g) < \infty$ for all $f,g \in B(X,Y)$.

• Clearly, $\rho(f,g) \geq 0$, and $\rho(f,g) = 0 \iff \sup_{x \in X} d_Y(f(x),g(x)) = 0$

$\iff d_Y(f(x),g(x)) = 0 \;\forall\, x \in X \iff f(x) = g(x) \;\forall\, x \in X$

$\iff f = g$.

• $\rho(f,g) = \sup_{x \in X} d_Y(f(x),g(x)) = \sup_{x \in X} d_Y(g(x),f(x)) = \rho(g,f)$.

Proposition: ρ is a metric on $B(X,Y)$.

Proof (continued):

- If $f, g, h \in B(X,Y)$ then for any $x \in X$,

$$d_Y(f(x), g(x)) \leq d_Y(f(x), h(x)) + d_Y(h(x), g(x)),$$ so also

$$\rho(f,g) = \sup_{x \in X} d_Y(f(x), g(x)) \leq \sup_{x \in X} \left(d_Y(f(x), h(x)) + d_Y(h(x), g(x)) \right)$$

$$\leq \sup_{x \in X} d_Y(f(x), h(x)) + \sup_{x \in X} d_Y(h(x), g(x)) = \rho(f,h) + \rho(h,g)$$

Proposition: If (Y, d_Y) is complete then so is $(B(X,Y), \rho)$.

Proof: Let $\{f_n\}_n$ be Cauchy in $B(X,Y)$. Then $\{f_n(x)\}_n$ is Cauchy in Y for every $x \in X$, so it converges to some $f(x) \in Y$.

Let $\varepsilon > 0$, let $N \in \mathbb{N}$ so that $\rho(f_n, f_m) < \varepsilon$ whenever $n, m \geq N$. For any $x \in X$ and $n \geq N$ we therefore get

$$d_Y(f(x), f_n(x)) = \lim_{m \to +\infty} d_Y(f_m(x), f_n(x)) \leq \varepsilon$$

$$\Longrightarrow \quad \rho(f, f_n) \leq \varepsilon.$$

$\blacksquare$

QUESTIONS?
COMMENTS?