CONVERGENCE OF EULER'S METHOD

Consider $(1) \begin{cases} y'(t) = f(y(t)) & (0 < t < T) \\ y(0) = \bar{y} \end{cases}$

where f is bounded, $|f(y)| \leq M \quad \forall\, y \in \mathbb{R}$.

Then also $|y'(t)| \leq M \quad \forall\, t$, so y is Lipschitz with constant M

If we approximate (1), then each approximation should also be Lipschitz with constant M

$\rightsquigarrow$ equicontinuity $\rightsquigarrow$ compactness

Consider (1) $\begin{cases} y'(t) = f(y(t)) & (0 < t < T) \\ y(0) = \bar{y} \end{cases}$

and <u>Euler's method</u> (2) $\begin{cases} \dfrac{y_{k+1} - y_k}{\Delta t} = f(y_k) & (k = 0, 1, \dots, N-1) \\ y_0 = \bar{y} \end{cases}$

where $\Delta t > 0$ is the <u>step size</u>, $y_k \approx y(t_k)$, $t_k = k\Delta t$, $N = T/\Delta t$ (so that $t_N = T$).

$$\boxed{\text{Does the method converge to } y \text{ as } \Delta t \longrightarrow 0\,?}$$

Recall: (1) $\iff$ $y(t) = \bar{y} + \int_0^t f(y(s))\,ds \quad \forall\ t \geq 0$ (1')

Similarly,

(2) $\iff$ $\begin{cases} y_{k+1} = y_k + \Delta t\, f(y_k) \\ y_0 = \bar{y} \end{cases} \quad \forall\ k = 0, 1, \ldots$

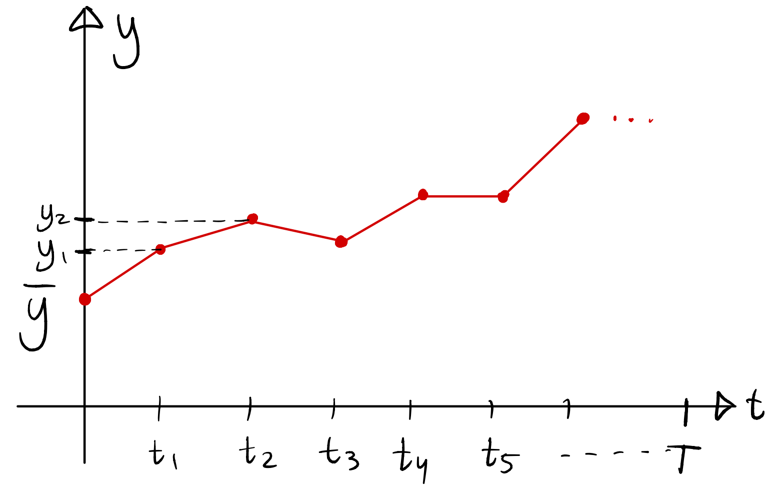
$\iff$ $y_k = \bar{y} + \underbrace{\Delta t \sum_{\ell=0}^{k-1} f(y_\ell)}_{\text{a "Riemann sum"}} \quad \forall\ k = 0, 1, \ldots$ (2')

$$y_k = \bar{y} + \Delta t \sum_{\ell=0}^{k-1} f(y_\ell) \qquad \forall \; k = 0, 1, \ldots \qquad (2')$$

We turn $\{y_0, y_1, \ldots\}$ into a <u>function</u> by defining

$$y^{\Delta t}(t) = y_k + \frac{t - t_k}{\Delta t}(y_{k+1} - y_k)$$

for $t \in [t_k, t_{k+1})$.



<u>Note</u>:

- $y^{\Delta t}(t_k) = y_k \qquad \forall \; k$
- $\frac{d}{dt} y^{\Delta t}(t\cancel{_k}) = f(y_k) \quad \forall \; t \in (t_k, t_{k+1})$
- $y^{\Delta t} \in C([0,T])$

Def.: Let (X, d) be a metric space. A subset $K \subseteq X$ is <u>relatively compact</u> if $\overline{K}$ is compact.

Prop.: Let K be relatively compact and $\{x_n\}_n$ a sequence in K. Then there is a subsequence $\{x_{n_k}\}_k$ and some $x \in X$ so that $x_{n_k} \xrightarrow[k \to \infty]{} x$.

Proof: $\{x_n\}_n$ is also a sequence in $\overline{K}$, which is compact, so there is a subseq. $\{x_{n_k}\}_k$ and some $x \in \overline{K}$ so that $x_{n_k} \xrightarrow[k \to \infty]{} x$. $\blacksquare$

The Arzela-Ascoli theorem:

Let (X,d) be compact and let $\mathcal{F} \subseteq C(X, \mathbb{R}^n)$. Then TFAE:

(i) $\mathcal{F}$ is compact

(ii) $\mathcal{F}$ is closed, bounded and equicontinuous.

Lemma (compactness): Assume that $f \in C_b(\mathbb{R})$ and let $\mathcal{F} = \{ y^{\Delta t} : 0 < \Delta t < 1 \}$. Then $\mathcal{F}$ is a relatively compact subset of $C_b([0,T])$.

Proof: • Claim: $\mathcal{F} \subseteq C_b([0,T])$.

Let $M = \sup_{x \in \mathbb{R}} |f(x)| < \infty$. If $t \in [t_k, t_{k+1})$ then

$$y^{\Delta t}(t) = y_k + \frac{t - t_k}{\Delta t}(y_{k+1} - y_k) = \bar{y} + \Delta t \sum_{\ell=0}^{k-1} f(y_\ell) + (t - t_k) f(y_k)$$

so

$$|y^{\Delta t}(t)| \leq |\bar{y}| + \Delta t \sum_{\ell=0}^{k-1} |f(y_\ell)| + |t - t_k| |f(y_k)|$$

$$\leq |\bar{y}| + \Delta t\, k M + \Delta t\, M \leq |\bar{y}| + M \Delta t N = |\bar{y}| + MT.$$

Hence, $\sup_{t \in [0,T]} |y^{\Delta t}(t)| \leq |\bar{y}| + MT$, so $y^{\Delta t} \in C_b([0,T])$.

Lemma (compactness): Assume that $f \in C_b(\mathbb{R})$ and let $\mathcal{F} = \{ y^{\Delta t} : 0 < \Delta t < 1 \}$. Then $\mathcal{F}$ is a relatively compact subset of $C_b([0,T])$.

- Claim: $\mathcal{F}$ is bounded.

Since $\sup_{t \in [0,T]} |y^{\Delta t}(t)| \leq |\bar{y}| + MT$ for every Δt, we get

$$\mathcal{F} \subseteq B(0; \, |\bar{y}| + MT + 1)$$

so $\mathcal{F}$ is bounded.

<u>Lemma (compactness)</u>: Assume that $f \in C_b(\mathbb{R})$ and let $\mathcal{F} = \{ y^{\Delta t} : 0 < \Delta t < 1 \}$. Then $\mathcal{F}$ is a relatively compact subset of $C_b([0,T])$.

- <u>Claim</u>: $\mathcal{F}$ is equicontinuous.

We have $\frac{d}{dt} y^{\Delta t}(t) = f(y_k)$ for $t \in (t_k, t_{k+1})$, so $\left| \frac{d}{dt} y^{\Delta t}(t) \right| \leq M$ for all t. Hence, all $y^{\Delta t}$ are Lipschitz with constant M. It follows that $\mathcal{F}$ is equicontinuous.

Lemma (compactness): Assume that $f \in C_b(\mathbb{R})$ and let $\mathcal{F} = \{ y^{\Delta t} : 0 < \Delta t < 1 \}$. Then $\mathcal{F}$ is a relatively compact subset of $C_b([0,T])$.

We have shown that $\mathcal{F} \subseteq C_b([0,T])$ is bounded and equicontinuous. Then $\overline{\mathcal{F}}$ is also bounded and equicontinuous (exercise!) and, of course, closed. Hence, by Arzela-Ascoli, $\overline{\mathcal{F}}$ is compact.

$\blacksquare$

Theorem: Let $f \in C_b(\mathbb{R})$. Then there is a subsequence $\{\Delta t_m\}_{m \in \mathbb{N}}$ such that $\Delta t_m \xrightarrow[m \to \infty]{} 0$ and $y^{\Delta t_m} \xrightarrow[m \to \infty]{} y$ uniformly, where y solves (1).

Proof: Let $\widetilde{\Delta t}_N = T/N$. Then $\widetilde{\Delta t}_N \xrightarrow[N \to \infty]{} 0$. Moreover, $\{y^{\widetilde{\Delta t}_N}\}_{N \in \mathbb{N}}$ is a sequence in the relatively compact set $\mathcal{F}$, so there is some subsequence $\{\Delta t_m\}_{m \in \mathbb{N}}$ of $\{\widetilde{\Delta t}_N\}_{N \in \mathbb{N}}$ so that $\{y^{\Delta t_m}\}_m$ converges to some $y \in C([0,T])$.

Theorem: Let $f \in C_b(\mathbb{R})$. Then there is a subsequence $\{\Delta t_m\}_{m \in \mathbb{N}}$ such that $\Delta t_m \xrightarrow[m \to \infty]{} 0$ and $y^{\Delta t_m} \xrightarrow[m \to \infty]{} y$ uniformly, where y solves (1).

Write $\hat{y}^{\Delta t}(t) = y_k$ for $t \in [t_k, t_{k+1})$. Then

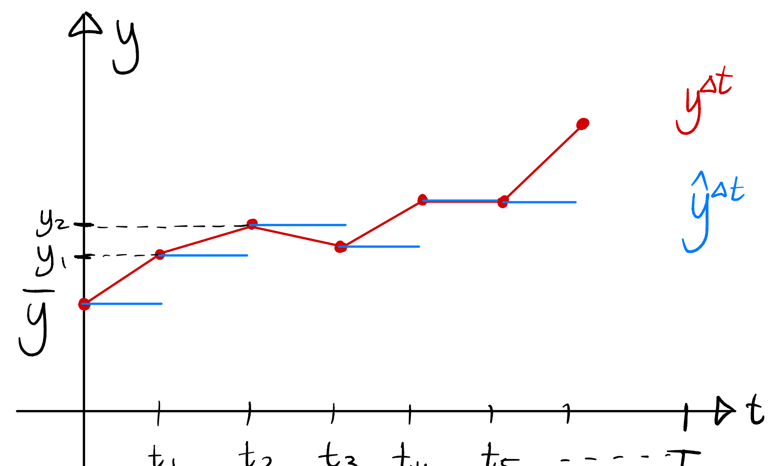
$$|y^{\Delta t}(t) - \hat{y}^{\Delta t}(t)| = \left| \frac{t - t_k}{\Delta t} (y_{k+1} - y_k) \right|$$

$$= |t - t_k| \, |f(y_k)|$$

$$\leq \Delta t \, M \xrightarrow[\Delta t \to 0]{} 0$$

so $\rho(y^{\Delta t}, \hat{y}^{\Delta t}) \xrightarrow[\Delta t \to 0]{} 0$, hence

also $\rho(\hat{y}^{\Delta t_m}, y) \xrightarrow[m \to \infty]{} 0$.



Theorem: Let $f \in C_b(\mathbb{R})$. Then there is a subsequence $\{\Delta t_m\}_{m\in\mathbb{N}}$ such that $\Delta t_m \xrightarrow[m\to\infty]{} 0$ and $y^{\Delta t_m} \xrightarrow[m\to\infty]{} y$ uniformly, where y solves (1).

Claim: y solves (1), that is, $y(t) = \bar{y} + \int_0^t f(y(s))\,ds \quad \forall\, t \in [0,T]$.

We have $y^{\Delta t}(t) = \bar{y} + \Delta t \sum_{\ell=0}^{k-1} f(y_\ell) + (t - t_k) f(y_k)$ for $t \in [t_k, t_{k+1})$.

Since $\Delta t\, f(y_\ell) = \int_{t_\ell}^{t_{\ell+1}} f(\hat{y}^{\Delta t}(s))\,ds$, we can write

$$y^{\Delta t}(t) = \bar{y} + \sum_{\ell=0}^{k-1} \int_{t_\ell}^{t_{\ell+1}} f(\hat{y}^{\Delta t}(s))\,ds + \int_{t_k}^{t} f(\hat{y}^{\Delta t}(s))\,ds$$

$$= \bar{y} + \int_0^t f(\hat{y}^{\Delta t}(s))\,ds.$$

Theorem: Let $f \in C_b(\mathbb{R})$. Then there is a subsequence $\{\Delta t_m\}_{m \in \mathbb{N}}$ such that $\Delta t_m \xrightarrow[m\to\infty]{} 0$ and $y^{\Delta t_m} \xrightarrow[m\to\infty]{} y$ uniformly, where y solves (1).

$$\Rightarrow \left| y(t) - \bar{y} - \int_0^t f(y(s))\,ds \right| \leq \left| y(t) - y^{\Delta t_m}(t) \right| + \left| y^{\Delta t_m}(t) - \left(\bar{y} + \int_0^t f(y(s))\,ds \right) \right|$$

$$= \left| y(t) - y^{\Delta t_m}(t) \right| + \left| \int_0^t f(\hat{y}^{\Delta t_m}(s))\,ds - \int_0^t f(y(s))\,ds \right| \xrightarrow[m\to\infty]{} 0$$

since (i) $\rho(y, y^{\Delta t_m}) \xrightarrow[m\to\infty]{} 0$,

(ii) $\rho(f \circ \hat{y}^{\Delta t_m}, f \circ y) \xrightarrow[m\to\infty]{} 0$, since $f \in C_b$ and $\rho(\hat{y}^{\Delta t_m}, y) \xrightarrow[m\to\infty]{} 0$.

$\blacksquare$

If we can prove, by some means, that the solution is unique, then we get convergence of the *entire* sequence.

Corollary: Let $f \in C_b(\mathbb{R})$ and assume that there exists at most one solution y of (1). Then $y^{\Delta t} \xrightarrow[\Delta t \to 0]{} y$.

Proof: If not, then there would be some subsequence $\{y^{\widetilde{\Delta t}_m}\}_{m \in \mathbb{N}}$ of $\{y^{\Delta t}\}_{\Delta t > 0}$ and some $\varepsilon > 0$ so that $\widetilde{\Delta t}_m \xrightarrow[m \to \infty]{} 0$ but $\rho(y^{\widetilde{\Delta t}_m}, y) \geq \varepsilon$ $\forall\, m \in \mathbb{N}$.

But we have just shown that $\{y^{\widetilde{\Delta t}_m}\}_m$ has some subsequence $\{y^{\Delta t_m}\}_m$ converging to some solution z of (1).

The solution is unique, so $z = y$, but then

$$0 < \varepsilon \leq \rho(y^{\Delta t_m}, y) = \rho(y^{\Delta t_m}, z) \xrightarrow[m \to \infty]{} 0 \quad ↯$$

$\blacksquare$

QUESTIONS?
COMMENTS?