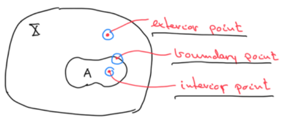
Section 3.3. Open and closed sets

In $\mathbb{R}$: (a,b) open
$[a,b]$ closed
$[a,b)$ neither open nor closed



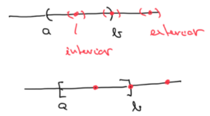
Definition: Assume that $A \subseteq X$. A point $x \in X$ is called

(i) an <u>interior point</u> of A if there is $r > 0$ such that $B(x;r) \subseteq A$.

(ii) an <u>exterior point</u> of A if there is $r > 0$ such that $B(x;r) \subseteq A^c$.

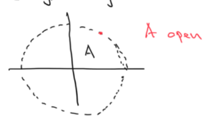
(iii) a <u>boundary point</u> if all balls $B(x;r)$ contain points from both A and A^c

Example: In $\mathbb{R}$

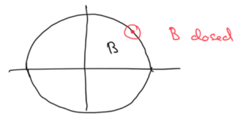


Definition: A set $A \subseteq X$ is <u>open</u> if it does not contain any of its boundary points. It is called <u>closed</u> if it contains <u>all</u> its boundary points.

Examples: $X = \mathbb{R}^2$ $A = \{(x,y) : x^2 + y^2 < 1\}$



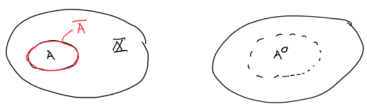
$B = \{(x,y) : x^2 + y^2 \leq 1\}$



Propoposition: A is open if and only if A^c is closed.
A is closed if and only if A^c is open.

Proof: A and A^c have the same boundary points, so if one of the sets has no boundary points, the other one must have them all.

Definition: Assume that $A \subseteq X$. Then the set ∂A is the set of all boundary points. Then $\overline{A} = A \cup \partial A$ is called the <u>closure</u> of A, and $A^\circ = A \setminus \partial A$ is called the <u>interior</u> of A.



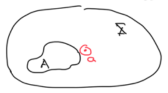
Prop: $\overline{A}$ is closed and A° is open.

Theorem: For a subset A of a metric space X, the following are equivalent:

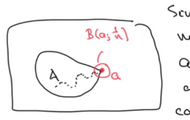
(i) A is closed

(ii) For all convergent sequences $\{a_n\}$ of elements in A, the limit $a = \lim_{n\to\infty} a_n$ is also in A.

Proof: (i) $\Rightarrow$ (ii) Assume that A is closed and that $\{a_n\}$ is a sequence of elements in A converging to a. We need to prove that $a \in A$. Assume for contradiction that <u>$a \notin A$</u>. Since A is closed, a has to be an exterior point. Since a is exterior, there is $\varepsilon > 0$ such that $B(a;\varepsilon) \subseteq A^c$. But then $a_n \notin B(a;\varepsilon)$, hence $a_n \not\to a$. This is a contradiction, and hence $a \in A$.



(ii) $\Rightarrow$ (i) We'll prove this contrapositively by showing that if A is <u>not</u> closed there is a sequence $\{a_n\}$ of points in A that converges to a point a that is not A.



Since A is not closed, there must be a boundary point a which is <u>not</u> in A. For all n, the ball <u>$B(a;\frac{1}{n})$</u> contains at least one element <u>a_n from A</u>. Pick one such element for each n, and consider the sequence $\{a_n\}$. Then $a_n \in A$, but $\lim a_n = a \notin A$.