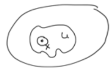
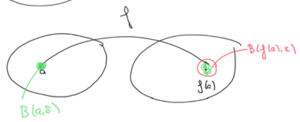
Section 3.3: Continuity in terms of open and closed sets

A neighborhood of a point $x \in X$ is just an open set U which contains x.



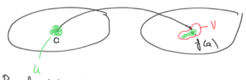


For every $\varepsilon > 0$, there is a $\delta > 0$ such that $f(B(a;\delta)) \subseteq B(f(a);\varepsilon)$.

Theorem: Assume $f: X \to Y$ is a function between two metric space and that $a \in X$. Then the following are equivalent.

(i) f is continuous at a

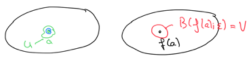
(ii) For every neighborhood V of $f(a)$, there is a neighborhood U of a such that $f(U) \subseteq V$.



Proof: (i) $\Rightarrow$ (ii) Assume that f is continuous at a and that V is a neighborhood of $f(a)$. We must find a neighborhood U of a such that $f(U) \subseteq V$.

Since V is a neighborhood of $f(a)$, there is a ball $B(f(a);\varepsilon) \subseteq V$. Since f is continuous at a $\delta > 0$ such that $f(B(a;\delta)) \subseteq B(f(a;\varepsilon)) \subseteq V$. If we put $U = B(a;\delta)$, we have found a neighborhood U of a such that $f(U) \subseteq V$.

(ii) $\Rightarrow$ (i) Assume that for ~~each~~ all neighborhoods V of a, there is neighborhood U of a such that $f(U) \subseteq V$. Given an $\varepsilon > 0$, we have to find a $\delta > 0$ such that $f(B(a;\delta)) \subseteq B(f(a);\varepsilon)$. Put $V = B(f(a;\varepsilon))$, then there is a neighborhood U of a such that $f(U) \subseteq V = B(f(a);\varepsilon)$.



Since U is a neighborhood of a, there is a ball $B(a;\delta)$ around a contained in U. But then

$$f(B(a;\delta)) \subseteq f(U) \subseteq V = B(f(a),\varepsilon)$$

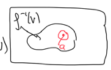
Hence f is continuous at a.

Theorem: Assume that $f: X \to Y$ is a function between two metric spaces. The following are equivalent:

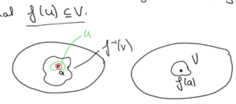
(i) f is continuous (at all points in X)

(ii) $f^{-1}(V)$ is open for all open sets V in Y.

Proof: (i) $\Rightarrow$ (ii) Assume that f is continuous and that $V \subseteq Y$ is open. We must prove that $f^{-1}(V)$ is open. Assume $a \in f^{-1}(V)$; we must find a δ such that $B(a;\delta) \subseteq f^{-1}(V)$



Obviously $f(a) \in V$, hence V is a neighborhood of $f(a)$. By the previous theorem, there is a neighborhood U of a such that $f(U) \subseteq V$.



This means that $U \subseteq f^{-1}(V)$. Since U is a neighborhood of a, there is a ball $B(a;\delta) \subseteq U \subseteq f^{-1}(V)$. Hence a is a interior point of $f^{-1}(V)$, and consequently $f^{-1}(V)$ is open.

(ii) $\Rightarrow$ (i) Assume now that $f^{-1}(V)$ is open for all open sets V in Y. We need to prove that f is continuous at any point $a \in X$. Using the previous theorem, it suffices to show that if V is a neighborhood of $f(a)$, then there is a neighborhood U of a such that $f(U) \subseteq V$. Put $U = f^{-1}(V)$, this is neighborhood of a and $f(U) = f(f^{-1}(V)) \subseteq V$.

Theorem: Assume that $f: X \to Y$ is a function between metric spaces. Then the following are equivalent:

(i) f is continuous

(ii) $f^{-1}(F)$ is closed for closed sets F.

Proof: (i) $\Rightarrow$ (ii) Assume that f is continuous and that F is closed. Must show that $f^{-1}(F)$ is closed. Since F is closed, F^c is open, and by the previous result

$$\underline{f^{-1}(F^c)} = \underline{(f^{-1}(F))^c} \quad \text{is open}$$

Hence $f^{-1}(F)$ is closed.

(ii) Assume that $f^{-1}(F)$ is closed when F is closed. It suffices to prove that for any open set U, $f^{-1}(U)$ is open. But if U is open, U^c is closed and hence

$$f^{-1}(U^c) = (f^{-1}(U))^c \quad \text{is closed}$$

This means that $f^{-1}(U)$ is open.