Section 3.3: Boolean operations with open and closed sets

Theorem: a) If $\mathcal{G}$ is a family of open sets, then $\bigcup_{G \in \mathcal{G}} G$ is also open.

b) If $G_1, G_2, \dots, G_n$ are open sets, then $G_1 \cap G_2 \cap \dots \cap G_n$ is also open.

Proof: a) Assume $x \in \bigcup_{G \in \mathcal{G}} G$. Then is at least one $G \in \mathcal{G}$ such that $x \in G$, and since G is open, there is a ball $B(x;\varepsilon) \subseteq G$. But then $B(x;\varepsilon) \subseteq \bigcup_{G \in \mathcal{G}} G$, hence x is an interior point in $\bigcup_{G \in \mathcal{G}} G$, and consequently $\bigcup_{G \in \mathcal{G}} G$ is open.

b) Assume $x \in G_1 \cap G_2 \cap \dots \cap G_n$; we must find an $r > 0$ such that $B(x;r) \subseteq G_1 \cap \dots \cap G_n$.

Since $x \in G_1 \cap G_2 \cap \dots \cap G_n$, we have $x \in G_1, x \in G_2, \dots, x \in G_n$. Since the sets are open, there are $r_1 > 0, r_2 > 0, \dots, r_n > 0$ such that $B(x;r_1) \subseteq G_1$, $B(x;r_2) \subseteq G_2, \dots, B(x;r_n) \subseteq G_n$. Choose $r = \min\{r_1, r_2, \dots, r_n\}$; then $B(x;r) \subseteq G_1$, $B(x;r) \subseteq G_2, \dots, B(x;r) \subseteq G_n$. Thus

$B(x;r) \subseteq G_1 \cap G_2 \cap \dots \cap G_n$. Hence $G_1 \cap G_2 \cap \dots \cap G_n$ is open.

Theorem: a) If $\mathcal{F}$ is a family of closed sets, then $\bigcap_{F \in \mathcal{F}} F$ is also closed.

b) If $F_1, F_2, \dots, F_n$ are closed sets, then $F_1 \cup F_2 \cup \dots \cup F_2$ is also closed.

Proof: a) Since each $F \in \mathcal{F}$ is close, F^c is open. Hence by the previous theorem, $\bigcup_{F \in \mathcal{F}} F^c$ is open. But by De Morgan's laws

$\bigcup_{F \in \mathcal{F}} F^c = \left(\bigcap_{F \in \mathcal{F}} F\right)^c$, and hence $\bigcap_{F \in \mathcal{F}} F$ is closed since its complement is open.

b) Since $F_1, F_2, \dots, F_n$ are closed, $F_1^c, F_2^c, \dots, F_n^c$ are open and hence

$F_1^c \cap F_2^c \cap \dots \cap F_n^c = (F_1 \cup F_2 \cup \dots \cup F_n)^c$ is open

But then $F_1 \cup F_2 \cup \dots \cup F_n$ is closed as it has an open complement.