Finding derivatives

Assume that we suspect that a bounded linear map $A: U \to V$ is the derivative of F at $\vec{a}$. To check this, we need to prove that

$$\sigma(\vec{r}) = \vec{F}(\vec{a}+\vec{r}) - \vec{F}(\vec{a}) - A(\vec{r})$$

goes to 0 faster than $\vec{r}$.

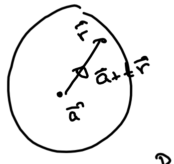
How to find A?

Definition: The directional derivative of $\vec{F}$ at $\vec{a}$ in the direction $\vec{r}$ is defined as

$$\vec{F}'(\vec{a};\vec{r}) = \lim_{t \to 0} \frac{\vec{F}(\vec{a}+t\vec{r}) - \vec{F}(\vec{a})}{t}$$

when the limit exists.

Compare $\vec{F}(\vec{a})(\vec{r})$



Proposition: If $\vec{F}$ is differentiable at $\vec{a}$, then

$$\vec{F}'(\vec{a};\vec{r}) = \vec{F}'(\vec{a})(\vec{r})$$

Proof: We have

$$\vec{F}(\vec{a}+\vec{r}) - \vec{F}(\vec{a}) = \vec{F}'(\vec{a})(\vec{r}) + \sigma(\vec{r})$$

$$\vec{F}'(\vec{a};\vec{r}) = \lim_{t \to 0} \frac{\vec{F}(\vec{a}+t\vec{r}) - \vec{F}(\vec{a})}{t}$$

$$= \lim_{t \to 0} \frac{\vec{F}'(\vec{a})(t\vec{r}) + \sigma(t\vec{r})}{t}$$

$$= \lim_{t \to 0} \left[\vec{F}'(\vec{a})(\vec{r}) + \frac{\sigma(t\vec{r})}{t\|\vec{r}\|}\|\vec{r}\| \right]$$

0

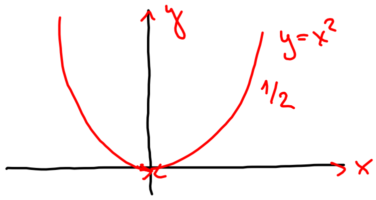
$$= \vec{F}'(\vec{a})(\vec{r}) + 0 = \vec{F}'(\vec{a})(\vec{r})$$

Warning: It is possible that all directional derivatives exist, but that the function is not differentiable (it may even fail to be continuous)

Reminder: $f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0). \end{cases}$$

All directional derivatives exist, but the function is not even continuous at 0



Method for finding derivatives:

(i) Compute the directional derivatives
$$\vec{F}'(\vec{a},\vec{r}) = \lim_{t\to 0} \frac{F(\vec{a}+t\vec{r})-\vec{F}(\vec{a})}{t}$$
This mean that the ~~linear~~ operator $A(\vec{r}) = \vec{F}'(\vec{a};\vec{r})$ is a candidate for $\vec{F}'(\vec{a})$.

(ii) Check that $A(\vec{r}) = \vec{F}'(\vec{a};\vec{r})$ is a bounded linear operator.

(iii) Check that
$$\sigma(\vec{r}) = \vec{F}(\vec{a}+\vec{r}) - \vec{F}(\vec{a}) - A(\vec{r})$$
goes to 0 faster than $\vec{r}$. Then $\vec{F}'(\vec{a}) = A$.

Example: Let $F : C([0,1],\mathbb{R}) \to \mathbb{R}$ be given by
$$\vec{F}(y) = y(0)^2$$
$$\uparrow$$
$$C([0,1],\mathbb{R})$$
Want to find $\vec{F}'(y)$.

(i) Find the directional derivatives
$$\vec{F}'(y;z) = \lim_{t\to 0} \frac{\vec{F}(y+tz)-\vec{F}(y)}{t}$$
$$= \lim_{t\to 0} \frac{(y(0)+tz(0))^2 - y(0)^2}{t}$$
$$= \lim_{t\to 0} \frac{\cancel{y(0)^2} + 2y(0)\cancel{t}z(0) + t^{\cancel{2}}z(0)^2 - \cancel{y(0)^2}}{\cancel{t}}$$
$$= \lim_{t\to 0} [2y(0)z(0) + tz(0)^2] = 2y(0)z(0)$$
Hence $F'(y;z) = 2y(0)z(0)$ and thus the candidate for the derivative at y is $\vec{F}'(y)(z) = 2y(0)z(0) = A(z)$

(ii) Need to check

A linear: $A(\alpha u+\beta v) = 2y(0)(\alpha u(0)+\beta v(0)) = \alpha 2y(0)u(0) + \beta 2y(0)v(0)$
$$= \alpha A(u) + \beta A(v)$$
A bounded:
$$\|A(z)\| = |2y(0)z(0)| \le \underbrace{2|y_0|}_{K}\|z\| = K\|z\|$$

(iii) Remains to check that A really is the derivative of F at y
$$\sigma(z) = \vec{F}(y+z) - \vec{F}(y) - A(z)$$
Need to prove that $\frac{\|\sigma(z)\|}{\|z\|} \to 0$ when $\|z\| \to 0$

$$\sigma(z) = (y(0)+z(0))^2 - y(0)^2 - 2y(0)z(0) \qquad A(z) = 2y(0)z(0)$$
$$= \cancel{y(0)^2} + \cancel{2y(0)z(0)} + z(0)^2 - \cancel{y(0)^2} - \cancel{2y(0)z(0)}$$
$$= z(0)^2 \le \|z\|^2$$
Thus
$$\frac{\|\sigma(z)\|}{\|z\|} \le \frac{\|z\|^2}{\|z\|} = \|z\| \to 0 \text{ as } \|z\| \to 0.$$
Hence we have proved that $\vec{F}'(y)(z) = 2y(0)z(0)$

Mean Value Theorem

In $\mathbb{R}$: $\dfrac{f(b)-f(a)}{b-a} = f'(c)$ for some $c \in (a,b)$

$$f(b)-f(a) = f'(c)(b-a)$$

Mean Value Theorem: Assume that V is a normed space and that $a, b \in \mathbb{R}$, $a < b$. Assume also that $\vec{F}: [a,b] \to V$ and $g: [a,b] \to \mathbb{R}$ are continuous function differentiable in (a,b) with $\|\vec{F}'(t)\| \leq g'(t)$ at all $t \in (a,b)$. Then

$$(*) \quad \|\vec{F}(b) - \vec{F}(a)\| \leq g(b) - g(a)$$

Proof: Given $\varepsilon > 0$, we shall prove that for all $t \in [a,b]$, we have

$$(**) \quad \|\vec{F}(t) - \vec{F}(a)\| \leq g(t) - g(a) + \varepsilon + \varepsilon(t-a)$$

In particular, we will have (putting $t=b$)

$$\|\vec{F}(b) - \vec{F}(a)\| \leq g(b) - g(a) + \varepsilon + \varepsilon(b-a) \text{ for any } \varepsilon > 0.$$

does not hold for all $t \in [a,b]$:

which implies $(*)$.

Assume for contradiction that $(**)$

Then

$$C = \{ t \in [a,b] : \|\vec{F}(t) - \vec{F}(a)\| > g(t) - g(a) + \varepsilon + \varepsilon(t-a) \}$$

is not empty, and let $c = \inf C$.

Note that: $c \neq a$, $c \neq b$.

Also $\|\vec{F}(c) - \vec{F}(a)\| = g(c) - g(a) + \varepsilon + \varepsilon(c-a)$

There exists a $\delta > 0$ such that for all $t \in (c, c+\delta]$ we have

$$\frac{\|\vec{F}(t) - \vec{F}(c)\|}{t-c} \leq \|\vec{F}'(c)\| + \frac{\varepsilon}{2}$$

$$\frac{g(t) - g(c)}{t-c} \geq g'(c) - \frac{\varepsilon}{2}$$

Then

$$\frac{\|\vec{F}(t) - \vec{F}(c)\|}{b-c} \leq \|\vec{F}'(c)\| + \frac{\varepsilon}{2} \leq g'(c) + \frac{\varepsilon}{2} \leq \frac{g(t)-g(c)}{t-c} + \varepsilon$$

Multiply by $(t-c)$

$$\|\vec{F}(t) - \vec{F}(c)\| \leq g(t) - g(c) + \varepsilon(t-c)$$

Hence

$$\|\vec{F}(t) - \vec{F}(a)\| \leq \|\vec{F}(t) - \vec{F}(c)\| + \|\vec{F}(c) - \vec{F}(a)\|$$

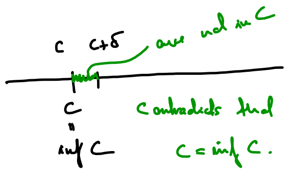
$$\leq g(t) - \cancel{g(c)} + \varepsilon(t-c) + \cancel{g(c)} - g(a) + \varepsilon + \varepsilon(c-a)$$

$$= g(t) - g(a) + \varepsilon + \varepsilon(t-a)$$

This means that for all $t \in (c, c+\delta]$, we have

$$\|\vec{F}(t) - \vec{F}(a)\| \leq g(t) - g(a) + \varepsilon + \varepsilon(t-a),$$

hence $t \notin C$.



Contradicts that $c = \inf C$.

This contradiction shows that C is empty and $(**)$ is proved.

Corollary: Assume $\vec{F}:[a,b]\to V$ is continuous and that $\|F'(t)\|\le k$ for all $t\in(a,b)$. Then

$$\|\vec{F}(b)-\vec{F}(a)\|\le k(b-a)$$

Proof: Let $g(t)=kt$. Then $g'(t)=k$ and hence

$$\|F'(t)\|\le g'(t)$$

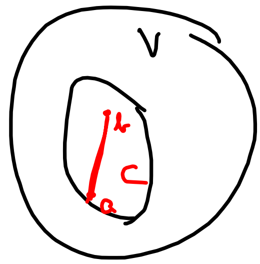
By MVT, we get

$$\|\vec{F}(b)-\vec{F}(a)\|\le g(b)-g(a)=kb-ka=k(b-a)$$

Def: A subset C of V is called convex if for all $\vec{a},\vec{b}\in C$ all the points

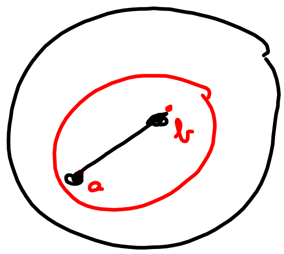
$$\vec{r}(t)=\vec{a}+t(\vec{b}-\vec{a}),\quad t\in[0,1]$$

also belong to C



Corollary: Assume that $C\subseteq U$ is convex and that the function $\vec{F}:U\to V$ is differentiable with $\|\vec{F}'(c)\|\le K$ for all $c\in C$. Then for all $\vec{a},\vec{b}\in C$, we have that

$$\|\vec{F}(\vec{b})-\vec{F}(\vec{a})\|\le K\|\vec{b}-\vec{a}\|$$



Proof: Let $H:[0,1]\to V$ is defined by

$$H(t)=\vec{F}(\vec{r}(t)),\quad t\in[0,1].$$

Chain Rules

$$H'(t)=\vec{F}'(\vec{r}(t))(\vec{r}'(t))=\vec{F}'(\vec{r}(t))(\vec{b}-\vec{a})$$

$$\vec{r}(t)=\vec{a}+(\vec{b}-\vec{a})t$$

$$\vec{r}'(t)=\vec{b}-\vec{a}$$

hence

$$\|H'(t)\|\le\|\vec{F}'(\vec{r}(t))\|\,\|\vec{b}-\vec{a}\|\le K\|\vec{b}-\vec{a}\|$$

By the Corollary above,

$$\begin{aligned}\|H(1)-H(0)\| &\le K\|\vec{b}-\vec{a}\|(1-0)=K\|\vec{b}-\vec{a}\| \\ \| \\ \|\vec{F}(\vec{b})-\vec{F}(\vec{a})\| &\end{aligned}$$